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Compactifying Moduli Spaces

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Foreword

This book comes from the lectures given during the conference "Compactifying Moduli Spaces", held in May 2013 at Centre de Recerca Matemàtica (CRM) in Barcelona. In them, the speakers reported on recent research on moduli theory, from different points of view.

In recent years, moduli spaces have been investigated for their diverse applications to Algebraic Geometry, Number Theory, String Theory, and Quantum Field Theory, just to mention a few. In particular, the notion of compactification of moduli spaces, in its various declinations, has played a crucial role in solving several open problems and long-standing conjectures.

The compactification problem can be approached via various techniques. Geometric Invariant Theory, Hodge Theory, and the MMP come into play as different approaches to construct and compactify moduli spaces. All these perspectives shed light on particular aspects of moduli problems.

In this volume, we collect three contributions, written by Radu Laza, Paul Hacking and Dragos Oprea. In the first of them, various ways to construct and compactify moduli spaces are presented. In the second, some questions on the boundary of moduli spaces of surfaces are addressed via stable vector bundles on the smooth surfaces which degenerate to boundary points. Finally, in the third contribution the theory of stable quotients is explained, which yields meaningful compactifications of moduli spaces of maps.

It is our hope that these lecture notes will illustrate the wide and rich variety of ideas and theories, which have developed from the general problem of understanding moduli spaces and their geometry.

Gilberto Bini
Martí Lahoz
Emanuele Macrì
Paolo Stellari

Chapter 1

Perspectives on the Construction and Compactification of Moduli Spaces

Radu Laza

Introduction

A central theme in algebraic geometry is the construction of compact moduli spaces with geometric meaning. The two early successes of the moduli theory – the construction and compactification of the moduli spaces of curves $\overline{M}_g$ and principally polarized abelian varieties (ppavs) $\overline{\mathcal{A}}_g$ – are models that we try to emulate. While very few other examples are so well understood, the tools developed to study other moduli spaces have led to new developments and unexpected directions in algebraic geometry. The purpose of these notes is to review three standard approaches to constructing and compactifying moduli spaces: GIT, Hodge theory, and MMP, and to discuss various connections between them.

One of the oldest approach to moduli problems is Geometric Invariant Theory (GIT). The idea is natural: the varieties in a given class can be typically embedded into a fixed projective space. Due to the existence of Hilbert schemes, one obtains a quasi-projective variety X parametrizing embedded varieties of a certain class. Forgetting the embedding amounts to considering the quotient X/G for a certain reductive algebraic group G. Ideally, X/G would be the moduli space of varieties of the given class. Unfortunately, the naive quotient X/G does not make sense; it has to be replaced by the GIT quotient $X/\!\!/G$ of Mumford [105]. While $X/\!\!/G$ is the correct quotient from an abstract point of view, there is a price to pay: it is typically difficult to understand which are the semistable objects (i.e., the objects parameterized by $X/\!\!/G$) and then some of the semistable objects are too degenerate from a moduli point of view. Nonetheless, $X/\!\!/G$ gives a projective model for a moduli space with weak modular meaning. Since the GIT model $X/\!\!/G$ is sometimes more accessible than other models, $X/\!\!/G$ can be viewed as a first approximation of more desirable compactifications of the moduli space.

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A different perspective on moduli is to consider the variation of the cohomology of the varieties in the given moduli stack $\mathcal{M}$. From this point of view, one considers the homogeneous space $\mathbb{D}$ that classifies the Hodge structures of a certain type, and then the quotient $\mathbb{D}/\Gamma$ which corresponds to forgetting the marking of the cohomology. The ideal situation would be a period map $\mathcal{P}\colon \mathcal{M} \to \mathbb{D}/\Gamma$ which is an isomorphism, or at least a birational map. Results establishing the (generic) injectivity of the period map are called "Torelli theorems", and a fair number of such results are known. Unfortunately, the image of $\mathcal{P}$ in $\mathbb{D}$ is typically very hard to understand: Griffiths' transversality says that the periods of algebraic varieties vary in a constrained way, which gives highly non-trivial systems of differential equations. Essentially, the only cases where we don't have to deal with these issues are the classical cases of ppavs and K3 surfaces, for which all our knowledge on their moduli is obtained by this Hodge theoretic construction. Furthermore, having a good period map gives numerous geometric consequences. The reason for this is that the spaces $\mathbb{D}/\Gamma$ have a lot of structure that can be translated into geometric properties. While it is advantageous to get a description of the moduli space as a locally symmetric variety, in practice very few examples are known. We will briefly mention some enlargement of the applicability of period map constructions to moduli beyond ppavs and K3s. Finally, we will review some work of Looijenga which gives some comparison results for the case when both the GIT and the Hodge theoretic approach are applicable. This is in some sense an ideal situation as both geometric and structural results exist.

While the first two approaches are based on considering the properties of smooth objects and then constructing a global moduli space, the third approach takes a different tack: one constructs a moduli space by gluing local patches. This gives a moduli stack, and the main issue is to carefully choose degenerations such that one obtains a proper and separated stack. By the valuative criteria, it suffices to consider 1-parameter degenerations. From a topological point of view, the ideal model is a semistable degeneration $\mathfrak{X}/\Delta$, but then the central fiber is far from unique. A fundamental insight comes from the minimal model program (MMP): the canonical model of varieties of general type is unique. Consequently, by allowing "mild" singularities, one obtains a unique limit for a 1-parameter degeneration, leading to a proper and separated moduli stack, and (under mild assumptions) even a projective coarse moduli space. This theory was developed by Kollár, Shepherd, Barron, and Alexeev (KSBA) with contributions from other authors. The relationship between the KSBA approach and the other two approaches is not well understood. We briefly review some partial results on this subject. In one direction, the connection between Du Bois and semi-log-canonical (slc) singularities gives a link between KSBA and Hodge theory. In a different direction, the Donaldson–Tian theory of K-stability establishes a connection between GIT and KSBA stability. These topics are rapidly evolving and suggest that much is still to be explored in moduli theory.

The overarching theme of these notes is that each approach sheds light on a different aspect of the moduli problem under consideration. By taking together different approaches one obtains a fuller picture of a moduli space and its compactifications. As examples of this principle, we point out to the theory of variation of GIT (VGIT) quotients of Thaddeus and of Dolgachev and Hu, and the study of log canonical models for $\overline{M}_g$, the so-called Hassett–Keel program.

Disclaimer. These notes reflect the interests and point of view of the author. We have tried to give a panoramic view of a number of topics in moduli theory and to point out some relevant references for further details. In particular, we point out the related surveys [27, 82, 91]. We apologize for any omissions and inaccuracies. For instance, there is no discussion of deformation theory [64, 112], stacks [119], or the log geometry point of view [1].

Acknowledgement. I have benefited from discussions with many people (including V. Alexeev, S. Casalaina-Martin, R. Friedman, P. Hacking, B. Hassett, E. Looijenga) over the years. I am particularly grateful to Y. Odaka and Z. Patakfalvi for some key comments on an earlier version of these notes.

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1.1 The GIT approach to constructing moduli spaces

Geometric Invariant Theory (GIT) is probably the most natural and classical approach to constructing moduli spaces. In this section, we will review some of the main points of the GIT approach and survey some applications of GIT to moduli. The standard reference for GIT is Mumford *et al.* [105]. Other good textbook references for GIT include [38, 102, 107]. For an extended discussion of the material included in this section see the survey [91].

1.1.1 Basic GIT and moduli

Many moduli spaces are naturally realized as quotients X/G, where X is some (quasi-)projective variety and G a reductive algebraic group acting on X. The following results lead to presentations of moduli spaces as quotients X/G:

(1) Given a class of polarized varieties, it is typically possible to give a uniform embedding result: *for all* (V, L) *in the given class, for* k *large and divisible enough* ($\textit{independently from } V$), L^k *is very ample and embeds* V *into a fix projective space* $\mathbb{P}^N$ (*with* $N = N(k)$ *independent from* V). For example:

Theorem 1.1.1 (Bombieri)**.** *For* V *a smooth surface with* K_V *big and nef, and for all* $k \geq 5$*, the linear system* $|kK_V|$ *gives a birational morphism* $V \to \overline{V} \subset \mathbb{P}^N$*, where* $\overline{V}$ *is the normal surface obtained by contracting all the smooth* (-2)*-curves orthogonal to* K_V*.*

Note that $\overline{V}$ is the canonical model of V and has at worst Du Val singularities.

Theorem 1.1.2 (Mayer)**.** *Let D be a big and nef divisor on a* K3 *surface. Then, for all $k \geq 3$, the linear system $|kD|$ gives a birational morphism $V \to \overline{V} \subset \mathbb{P}^N$, where $\overline{V}$ is the normal surface obtained by contracting all the smooth (-2)-curves orthogonal to D.*

While similar results for singular varieties and higher dimensions are more subtle, satisfactory (but non-effective) results do exist in high generality (e.g., see [2, 62]). In other words, we can assume without loss of generality that all varieties V in a certain class are embedded in a fixed projective space $\mathbb{P}^N$.

(2) There exists a *fine* moduli space for embedded schemes V in $\mathbb{P}^N$ with fixed numerical invariants (i.e., Hilbert polynomial $p_V(t)$): it is the Hilbert scheme $H := \mathrm{Hilb}_{p_V}(\mathbb{P}^N)$. This is a well-known story (e.g., [80, Ch. 1]), we only emphasize here the connection between flatness and the preservation of numerical invariants [63, Thm. III.9.9], and the fact that the Hilbert scheme is one of the very few instances of fine moduli spaces in algebraic geometry.

(3) To pass from the Hilbert scheme H to a moduli space, there are two final steps. First, the Hilbert scheme parameterizes many objects that might have no connection to the original moduli problem (e.g., entire components of H might parameterize strange non-reduced schemes). Thus, we need to restrict to the locus $X \subset H$ of "good" objects. To get a good theory, it is needed that X is a locally closed subvariety of H (e.g., the defining conditions for good objects are either open or closed conditions). Local closeness for the moduli functor holds quite generally, but sometimes it is quite subtle (e.g., [81]). Finally, to forget the embedding $V \subset \mathbb{P}^N$ amounts to allowing linear changes of coordinates on $\mathbb{P}^N$. In conclusion, by this construction, we essentially obtained a moduli space for varieties of a given class as a global quotient X/G, where $X \subset H$ is as above and $G = \mathrm{PGL}(N+1)$. (For technical reasons, we replace X by its closure and $\mathrm{PGL}(N+1)$ by $\mathrm{SL}(N+1)$ in what follows.)

The naive quotient X/G typically does not make sense. The correct solution is the GIT quotient $X/\!\!/G$. Ideally, we would like that:

(a) the quotient X/G gives a 1-to-1 parameterization of the G-orbits in X, and
(b) X/G has the structure of an algebraic variety (such that $X \to X/G$ is a morphism that is constant on orbits).

This is rarely possible, as the example of $G = \mathbb{C}^*$ acting in the standard way on $\mathbb{A}^1 = \mathbb{C}$ shows: there are two orbits, $\mathbb{A}^1 \setminus \{0\}$ and $\{0\}$, but they cannot give two separate points in X/G, as this would contradict the continuity of $X \to X/G$ (note that $\{0\} \subset \overline{\mathbb{A}^1 \setminus \{0\}} = \mathbb{A}^1$). The GIT solution is to relax condition (a) and then use (b) to define a quotient in a universal categorical sense. In the affine case, $X = \operatorname{Spec} R$, it is easy to see that there is only one possible choice

$$X/G := \operatorname{Spec} R^G,$$

where R^G is a ring of G-invariant regular functions (automatically a finitely generated algebra if G is reductive). With this definition, all the expected properties of $X \to X/G$ hold except for (a), which is replaced by

(a′) every point in X/G corresponds to a unique closed orbit (and two orbits map to the same point in X/G if and only if the intersection of their closures is non-empty).

For example, the invariant ring $\mathbb{C}[x]^{\mathbb{C}^*}$ (with $t \in \mathbb{C}^*$ acting by $x \mapsto tx$) is the ring of constants $\mathbb{C}$ and thus $\mathbb{A}^1/\mathbb{C}^* = \{*\}$ corresponding to two different orbits, one of which (i.e., $\{0\}$) is closed.

In general, a quotient X/G can be constructed by gluing quotients of open affine G-invariant neighborhoods of points in X. For simplicity, we restrict here to the case of X being a projective variety with an ample G-linearized line bundle $\mathcal{L}$. In this situation, the correct quotient from the algebraic geometry point of view is

$$X/\!\!/G := \operatorname{Proj} R(X, \mathcal{L})^G, \tag{1.1.1}$$

where $R(X, \mathcal{L}) = \bigoplus_n H^0(X, \mathcal{L}^n)$. Note that in the projective situation, the natural map $X \dashrightarrow X/\!\!/G$ is only a rational map: it is defined only for *semistable points* $x \in X^{\mathrm{ss}}$, i.e., points for which there exists an invariant section $\sigma \in H^0(X, \mathcal{L}^n)$ not vanishing at x. The stable locus $X^{\mathrm{s}} \subseteq X^{\mathrm{ss}}$ is the (open) set of semistable points x for which the orbit $G \cdot x$ is closed in X^{ss} and the stabilizer G_x is finite. The quotient X^s/G is a geometric quotient, i.e., satisfies both conditions (a) and (b), and thus a good outcome for a moduli problem.

At this point, we already see some issues with constructing a moduli space via GIT. First, the set of semistable points is somewhat mysterious and might not be what is expected. Secondly, if there exist strictly semistable points (i.e., if $X^{\mathrm{ss}} \setminus X^{\mathrm{s}} \neq \emptyset$), then several orbits will correspond to the same point in the quotient $X/\!\!/G$. Thus, usually, the GIT quotients are not "modular" at the boundary.

Remark 1.1.3. It is well known that in order to obtain finitely generated rings of invariants R^G it is essential to work with reductive groups G. However, there exist natural situations when G is not reductive. We point to [78] for some techniques to handle these cases. For some concrete examples of non-standard GIT (e.g., G non-reductive or non-ample linearization $\mathcal{L}$) see [29, 30].

The GIT quotient $X/\!\!/G$ depends on the choice of linearization. This gives flexibility to the GIT construction, which is sometimes very useful. By definition (cf., (1.1.1)), the GIT quotient depends on the choice of linearization $\mathcal{L}$, so it is more appropriate to write $X/\!\!/_{\mathcal{L}}G$. A surprising fact discovered by Dolgachev and Hu [39] and Thaddeus [123] is that the dependence on $\mathcal{L}$ is very well behaved (see [91, §3] for further discussion):

(1) There are finitely many possibilities for the GIT quotients $X/\!\!/_{\mathcal{L}}G$ as one varies the linearization $\mathcal{L}$. The set of linearizations is partitioned into rational polyhedral chambers parameterizing GIT equivalent linearizations.

(2) The semistable loci satisfy a semi-continuity property. This property induces morphisms between quotients for nearby linearizations.

(3) The birational change of the GIT quotient as the linearization moves from one chamber to another by passing a wall is flip like, and can be described quite explicitly.

The above properties lead to one of the main strengths of the theory of Variation of GIT quotients (VGIT): *it might be possible to interpolate from an easily understood space* $\mathcal{M}_0$ *to a geometrically relevant space* $\mathcal{M}_1$ *by varying the linearization in a* VGIT *set-up.* A spectacular application of this principle is Thaddeus' work [122] on the Verlinde formula. A more modest application (but closer to the spirit of these notes) of VGIT to moduli is Theorem 1.1.8 below.

Example 1.1.4. We discuss here a simple example of VGIT (for details see [91, §3.4]) which illustrates the interpolation between stability conditions as the linearization varies. Specifically, we consider GIT for pairs (C, L) consisting of a plane cubic and a line. In this situation, the G-linearizations are parameterized by $t \in \mathbb{Q}$, and they are effective (i.e., there exist invariant sections) for $t \in [0, \frac{3}{2}]$. The GIT stability for the pair (C, L) for parameter $t \in [0, \frac{3}{2}]$ is described as follows: *If L passes through a singular point of C, then the pair (C, L) is t-unstable for all $t > 0$. Otherwise, (C, L) is t-(semi)stable for an interval $t \in (\alpha, \beta)$ (resp. $t \in [\alpha, \beta]$), where*

$$\alpha = \begin{cases} 0 & \text{if } C \text{ has at worst nodes,} \\ \frac{3}{5} & \text{if } C \text{ has an } A_2 \text{ singularity,} \\ 1 & \text{if } C \text{ has an } A_3 \text{ singularity,} \\ \frac{3}{2} & \text{if } C \text{ has a } D_4 \text{ singularity,} \end{cases} \quad \text{and} \quad \beta = \begin{cases} \frac{3}{5} & \text{if } L \text{ is inflectional to } C, \\ 1 & \text{if } L \text{ is tangent to } C, \\ \frac{3}{2} & \text{if } L \text{ is transversal to } C. \end{cases}$$

In other words, for $t = 0$ the semistability of (C, L) is equivalent to the semistability of C; for $t = 3/2$ the semistability of (C, L) is equivalent to the semistability of $C \cap L \subset L \cong \mathbb{P}^1$. The intermediate stability conditions are interpolations of these two extremal conditions.

Some of the main tools of GIT are the numerical criterion, Luna's slice theorem, and Kirwan's desingularization. In general, it is very hard to describe the (semi)stability conditions for a GIT quotient $X /\!\!/ G$. Essentially, the only effective tool for this is the numerical criterion. The main points that lead to the numerical criterion are as follows. By definition, the (semi)stability of $x \in X$ is related to the study of orbit closures. By the valuative criterion of properness, to test that an orbit is closed it suffices to consider 1-parameter families with all fibers in the same orbit. The key point now is that, in the GIT situation, it suffices to study 1-parameter families induced by 1-parameter subgroups $\lambda(t)$, $t \in \mathbb{C}^*$. To such a family, one associates a numerical function $\mu^{\mathcal{L}}(x, \lambda)$ (where $\mathcal{L}$ is a G-linearization on X) which measures the limiting behavior as $t \to 0$. This leads to the following:

Theorem 1.1.5 (Hilbert–Mumford Numerical Criterion)**.** *Let $\mathcal{L}$ be an ample G-linearized line bundle. Then $x \in X$ is stable (resp. semistable) with respect to $\mathcal{L}$ if and only if $\mu^{\mathcal{L}}(x,\lambda) > 0$ (resp. $\mu^{\mathcal{L}}(x,\lambda) \geq 0$) for every nontrivial 1-PS λ of G.*

The numerical criterion splits the a priori intractable problem of deciding semistability (i.e., finding non-vanishing G-invariant sections σ at $x \in X$) into two somewhat accessible steps. The first one is a purely combinatorial step involving the weights of the maximal torus in G on a certain representation. Furthermore, it is possible to include the variation of linearization in this combinatorial analysis. This first step can be effectively solved with the help of a computer (see [91, §4.1]). The second step consists in interpreting geometrically the results of the first step. This is a case by case delicate analysis, and the true bottleneck for the wide applicability of GIT. In any case, for any concrete GIT problem, the numerical criterion gives an algorithmic approach to semistability (see §1.1.2 for a survey of the known GIT examples). In fact, it is reasonable to say that GIT is the most accessible/computable approach to moduli spaces.

By construction, the quotient $X /\!\!/_{\mathcal{L}} G$ is a normal projective variety. It is natural to ask about its local structure. Of course, the local structure at an orbit $G \cdot x \in X /\!\!/ G$ depends on the local structure of $x \in X$ and the stabilizer G_x. For instance, if X is smooth, then the geometric quotient X^s/G has only finite quotient singularities (or equivalently, it is a smooth Deligne–Mumford stack), and thus it is well behaved. In general, for $x \in X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ (and, without loss of generality, assuming that $G \cdot x$ is a closed orbit, and thus G_x is reductive by the Matsushima criterion) Luna's theorem gives a precise description of the local structure of the quotient near the orbit $G \cdot x \in X /\!\!/ G$:

Theorem 1.1.6 (Luna slice theorem)**.** *Given $x \in X^{\mathrm{ss}}$ with closed orbit $G \cdot x$ and X smooth, there exists a G_x-invariant normal slice $V_x \subset X^{\mathrm{ss}}$ (smooth and affine) to $G \cdot x$ such that we have the following commutative diagram with Cartesian squares:*

$$\begin{array}{ccccc} G *_{G_x} \mathcal{N}_x & \overset{\text{étale}}{\longleftarrow} & G *_{G_x} V_x & \overset{\text{étale}}{\longrightarrow} & X^{\mathrm{ss}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{N}_x / G_x & \overset{\text{étale}}{\longleftarrow} & (G *_{G_x} V_x)/G & \overset{\text{étale}}{\longrightarrow} & X /\!\!/ G, \end{array}$$

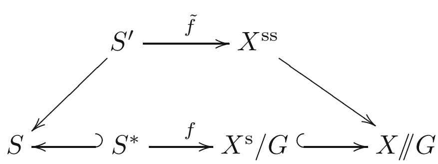
where $\mathcal{N}_x$ is the fiber at x of the normal bundle to the orbit $G \cdot x$.

The main point here is that this theorem reduces (locally) the GIT quotient $X /\!\!/ G$ to a GIT quotient by a smaller reductive group $G_x \subset G$ (typically G_x is a torus, or even $\mathbb{C}^*$). As an application of this, it is easy to resolve the GIT quotients $X /\!\!/ G$. Specifically, one orders the stabilizers G_x in the obvious way, and then blows-up the strata corresponding to the maximal stabilizers G_x. Roughly, and locally, this corresponds to the blow-up of the origin in $\mathcal{N}_x$. Since the action of G_x is distributed on the exceptional divisor of $\mathrm{Bl}_0 \mathcal{N}_x$, one sees that after such a blow-up the resulting stabilizers are strictly contained in G_x. Repeating the process inductively, one arrives to the ideal situation where all the stabilizers are finite. This resolution process was developed by Kirwan [77]. As an application, Kirwan used this desingularization process to compute the cohomology of several moduli spaces (e.g., [76]). As an aside, we note that Luna's theorem and Kirwan's desingularization procedure explain why various moduli spaces are nested. For instance, the exceptional divisor obtained by resolving the "worst" singularity for the GIT quotient for cubic fourfolds [88] is naturally identified with the moduli of degree 2 K3 surfaces (this is also related to the discussion of §1.2.3).

For a survey of the singularities of GIT quotients see [91, §5.1]. For a discussion of Luna's theorem in connection to VGIT see [91, §4.2].

The main advantage of GIT is that it gives projective models for moduli spaces with weak modular meaning. Specifically, by construction, a GIT quotient $X/\!\!/G$ is a normal projective variety (assuming, as above, X normal projective). Furthermore, each point of the quotient $X/\!\!/_{\mathcal{L}}G$ corresponds to a unique closed orbit. If there are no strictly semistable points, then $X/\!\!/G$ is also a geometric quotient, and thus the coarse moduli space of a proper Deligne–Mumford moduli stack. This is typically not the case: there exist strictly semistable points, and thus multiple orbits correspond to the same point in the quotient $X/\!\!/G$. It is (typically) not possible to define a functor that selects only the closed orbits. Nonetheless, the weak modular properties of GIT quotients might be the best that one can expect short of a DM stack (see Alper [12] for a formalization of "good" moduli stacks). Note also the following (a consequence of the properness of $X/\!\!/G$ and of the 1-to-1 correspondence between points in $X/\!\!/G$ and closed orbits):

Proposition 1.1.7 (GIT Semistable replacement lemma)**.** *Let* $S = \mathrm{Spec}(R)$ *and* $S^* = \mathrm{Spec}(K)$*, where* R *is a* DVR *with field of fractions* K *and closed point* o*. Assume that* $S^* \to X^{\mathrm{s}}/G$ *for some GIT quotient. Then, after a finite base change* $S' \to S$ (*ramified only at the special point* o)*, there exists a lift* $\tilde{f}\colon S' \to X^{\mathrm{ss}}$ *of* f *as in the diagram*



Furthermore, one can assume that $\tilde{f}(o)$ *belongs to a closed orbit.*

In other words, while a GIT quotient typically fails to have a modular meaning at the boundary, one can use this lemma to understand the degenerations of smooth objects and then construct or understand a good compactification of the moduli space. A concrete application of this principle is discussed in §1.2.3 below. For some further discussion and examples see [27, Ch. 11].

Unfortunately, GIT only sees "linear" features of the parameterized varieties. Consequently, quite degenerate objects might be semistable, leading to bad singularities for the GIT quotient $X/\!\!/G$ and the failure of modularity.

We have mentioned several drawbacks of the GIT approach to moduli spaces. It is difficult to decide stability of objects. Also, typically, there exist strictly semistable points. At these points, the GIT quotient is quite singular, and fails to be modular. The hidden geometric reason for these issues is that GIT only tests for linear features of the objects under consideration (e.g., see Remark 1.3.6). By considering asymptotic GIT (higher and higher embeddings of a given object), more of the geometric features of the varieties will be visible by means of "linear tests". Unfortunately, as discussed in §1.3.3, the asymptotic GIT approach is neither well behaved nor well understood.

The main point we want to emphasize here is that the unstable objects always satisfy some special conditions with respect to a flag of linear subspaces (e.g., an unstable hypersurface will always contain a singular point and there will be a special tangent direction through this point). This follows, for instance, from the work of Kempf [74]. Namely, for unstable points, there is a distinguished 1-PS λ that destabilizes x (essentially minimizing $\mu(x,\lambda)/|\lambda|$). Then, λ determines a parabolic subgroup P_λ, which in turn is equivalent to a (partial) flag. The failure of stability involves some special geometric properties with respect to this flag. Consequently, objects that behave well with respect to linear subspaces will tend to be semistable. For instance, a conic of multiplicity $d/2$ is semistable when viewed as a curve of degree d. Of course, such objects would be disallowed by other more "modular" approaches. For instance, for $d = 4$ (plane quartics, or genus 3 curves), the double conic should be replaced by hyperelliptic curves.

1.1.2 Applications of GIT to moduli

In this section, we briefly review the scope of GIT constructions in moduli theory.

Survey of GIT constructions in moduli theory. GIT and moduli were tightly connected for over a hundred years. Initially, in late 1800s and early 1900s, the focus was on computing explicitly the rings of invariants for various quotients X/G (for example, the ring of invariant polynomials for cubic surfaces). After Hilbert's proof of the finite generation of the ring of invariants R^G, the search of explicit invariants fell out of favor. Mumford [105] revived GIT to show that the moduli space of curves M_g is quasi-projective [105, Thm. 7.13]. The case of abelian varieties (with level structure) is also discussed in Mumford's monograph [105, Thm. 7.9]. A little later, Mumford and Gieseker [103] proved, via GIT, that the coarse moduli space $\overline{M}_g$ associated to the Deligne–Mumford compactification of the moduli space of genus g curves is a projective compactification of M_g (for a discussion of this and related constructions, see the survey [101]). Some other major results around the same time include the proof of quasi-projectivity for the moduli of surfaces of general type (Gieseker [54]) and compactifications for the moduli spaces of vector bundles over curves (Mumford, Narasimhan, Seshadri, e.g., [113]) and surfaces (Gieseker [55]). More recently, Viehweg [127] proved the quasi-projectivity of moduli of varieties of general type (see also 1.3.4 and 1.3.5 below) by using non-standard linearizations on the moduli space.

The GIT constructions for $\overline{M}_g$ or moduli of surfaces of general type involve *asymptotic* GIT, i.e., given a class of polarized varieties (V, L) one considers the GIT quotient of the Chow variety Chow_k (or Hilbert scheme) for higher and higher embeddings $V \to \mathbb{P}^{N(k)}$ given by L^k (for $k \gg 0$). In the case of curves, the quotients $\mathrm{Chow}_k /\!\!/ \mathrm{SL}(N_k)$ stabilize and give $\overline{M}_g$ (in fact, $k \geq 5$ is enough). For surfaces, as discussed in §1.3.3, there is no stabilization for the asymptotic GIT and it is unclear how to use this asymptotic approach to construct a compact moduli space.

In recent years, in connection to the Hassett–Keel program, there is a renewed interest in understanding non-asymptotic GIT models for $\overline{M}_g$. Other non-asymptotic GIT quotients that were studied include moduli for some hypersurfaces: plane sextics [115], quartic surfaces [116], cubic threefolds [9], cubic fourfolds [88], and some complete intersections (e.g., [17, 19, 30]). We emphasize that for hypersurfaces it is possible to give, in an algorithmic way, the shape of equations defining unstable hypersurfaces. However, it is difficult to interpret geometrically the stability conditions. In fact, the higher the degree, the worst the singularities that are allowed for stable objects. Consequently, GIT will give a somewhat random compactification for the moduli of hypersurfaces.

GIT and the Hassett–Keel program. It is of fundamental interest to understand the birational geometry of $\overline{M}_g$ (see the survey [44]), in particular the canonical model $\overline{M}_g^{\mathrm{can}}$ (for $g \geq 24$). A fundamental insight (due to Hassett and Keel) says that one can approach this problem via interpolation (see esp. [66, 67]). Namely, one defines

$$\overline{M}_g(\alpha) = \mathrm{Proj}(R(\overline{M}_g, K_{\overline{M}_g} + \alpha\Delta)),$$

where Δ is the boundary divisor in $\overline{M}_g$. For $\alpha = 1$ (and all g) one gets the Deligne–Mumford model $\overline{M}_g$, while for $g \geq 24$ and $\alpha = 0$ one gets $\overline{M}_g^{\mathrm{can}}$. It was observed that some of the $\overline{M}_g(\alpha)$ models have modular meaning. For instance, $\overline{M}_g(9/11)$ is a moduli of pseudo-stable curves, i.e., curves with nodes and cusps and without elliptic tails. It is conjectured that all $\overline{M}_g(\alpha)$ for $\alpha \in [0, 1]$ have some (weak) modular meaning, and that they behave similarly to the spaces in a VGIT set-up. In other words, conjecturally, via a finite number of explicit and geometrically meaningful modifications, one can pass from $\overline{M}_g = \overline{M}_g(1)$ to $\overline{M}_g^{\mathrm{can}} = \overline{M}_g(0)$. In this way, one would obtain a satisfactory description of the canonical model $\overline{M}_g^{\mathrm{can}}$ as well as a wealth of information on the birational geometry of $\overline{M}_g$.

Most of the constructed $\overline{M}_g(\alpha)$ spaces were obtained via GIT (see [14, 46] for some recent surveys). Roughly speaking, GIT tends to give compactifications with small boundary. Consequently, once a GIT model $\overline{M}_g^{\mathrm{GIT}}$ with the correct polarization was constructed, one can show that it agrees with $\overline{M}_g(\alpha)$ on open subsets with high codimension boundaries in $\overline{M}_g(\alpha)$ and $\overline{M}_g^{\mathrm{GIT}}$, respectively (and thus $\overline{M}_g(\alpha)$ and $\overline{M}_g^{\mathrm{GIT}}$ agree everywhere). The important point here is that $\overline{M}_g^{\mathrm{GIT}}$ is a projective variety and its polarization can be easily understood by descent from a parameter space (typically a Chow variety or a Hilbert scheme for small embeddings of genus g curves). Another important point is that $\overline{M}_g^{\mathrm{GIT}}$ comes with a (weak) modular interpretation and thus it induces such a modular interpretation for $\overline{M}_g(\alpha)$.

As already mentioned, the behavior of $\overline{M}_g(\alpha)$ seems to be parallel to that of quotients in a VGIT set-up. One might conjecture that there is a master VGIT problem modeling all of $\overline{M}_g(\alpha)$. Unfortunately, the best result so far is the following.

Theorem 1.1.8 ([30]). *For $\alpha \leq 5/9$, the log minimal models $\overline{M}_4(\alpha)$ arise as* GIT *quotients of the parameter space $\mathbb{P}E$ for $(2,3)$ complete intersections in $\mathbb{P}^3$. Moreover, the* VGIT *problem gives us the following diagram:*

$$\begin{array}{ccccccccc}
 & \overline{M}_4(\frac{5}{9},\frac{23}{44}) & & \dashrightarrow & \overline{M}_4(\frac{23}{44},\frac{1}{2}) & & \dashrightarrow & \overline{M}_4(\frac{1}{2},\frac{29}{60}) & \\
\swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \Searrow \\
\overline{M}_4(\frac{5}{9}) & & & \overline{M}_4(\frac{23}{44}) & & & \overline{M}_4(\frac{1}{2}) & & \overline{M}_4[\frac{29}{60},\frac{8}{17}) \\
 & & & & & & & & \downarrow \\
 & & & & & & & & \overline{M}_4(\frac{8}{17}) = \{*\}
\end{array} \tag{1.1.2}$$

More specifically,

(i) *the end point $\overline{M}_4(\frac{8}{17}+\epsilon)$ is obtained via* GIT *for $(3,3)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$, as discussed in* [45]*;*

(ii) *the other end point $\overline{M}_4(\frac{5}{9})$ is obtained via* GIT *for the Chow variety of genus 4 canonical curves, as discussed in* [29]*;*

(iii) *the remaining spaces $\overline{M}_4(\alpha)$ for α in the range $\frac{8}{17} < \alpha < \frac{5}{9}$ are obtained via appropriate $\mathrm{Hilb}^m_{4,1}$ quotients, with the exception of $\alpha = \frac{23}{44}$.*

The Hassett–Keel program is currently established for either values of α close to 1 (e.g., [66, 67]) or small values of the genus g (e.g., [29, 30, 45, 47, 70]). For some general predictions on $M_g(\alpha)$, see [13].

1.2 Moduli and periods

A different approach to the construction of moduli spaces is based on the idea of associating to a variety its cohomology, and studying the induced variation of Hodge structures (VHS). As we will discuss here, the scope of this approach is quite limited in practice. However, when the Hodge theoretic approach is applicable, it has strong implications on the structure of the moduli space; and thus this is a highly desirable situation. We discuss some examples of moduli space for which both the GIT and Hodge theoretic approaches are applicable. Each approach gives a different facet of the moduli space.

Some general references for the material discussed here include [26, 56, 99, 130] and [118, Ch. 2, 3].

1.2.1 Period maps

The primitive cohomology of a smooth projective variety carries a polarized Hodge structure such that the associated Hodge filtration varies holomorphically in families. More formally, one says that for a smooth family $\pi\colon \mathcal{X} \to S$ of algebraic varieties, $(R^n\pi_*\mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}}$ defines a polarized Variation of Hodge Structures (VHS) over S, which in turn defines a period map $\mathcal{P}\colon S \to \mathbf{D}/\Gamma$. In order to use $\mathcal{P}$ to construct a moduli space, we need to discuss the injectivity of $\mathcal{P}$ ("Torelli Theorems") and the image of $\mathcal{P}$.

Period domains and period maps. The period domain $\mathbf{D}$ is the classifying space of Hodge structures of a given type. Specifically, the polarized Hodge structures of weight n satisfy the Hodge–Riemann bilinear relations:

(HR1) $F^p = (F^{n-p+1})^\perp$;

(HR2) $(-1)^{n(n-1)} i^{p-q}(\alpha, \bar{\alpha}) > 0$ for $\alpha \in H^{p,q} = F^p \cap \overline{F^q}$ (with $p+q=n$).

The first condition (HR1) defines a projective homogeneous variety $\check{\mathbf{D}} = \mathbb{G}_{\mathbb{C}}/\mathbf{B}$, a subvariety of a flag manifold. Condition (HR2) gives that $\mathbf{D}$ is an open subset (in the classical topology) of $\check{\mathbf{D}}$, in particular a complex manifold. Note also that the period domain is homogeneous: $\mathbf{D} = \mathbb{G}_{\mathbb{R}}/\mathbf{K}$ (with $\mathbf{K} = \mathbf{B} \cap \mathbb{G}_{\mathbb{R}}$ a compact subgroup of $\mathbb{G}_{\mathbb{R}}$), and *semi-algebraic* (given by algebraic inequalities involving Re and Im of holomorphic coordinates). It is important to note that there are only two cases when $\mathbf{D}$ is a Hermitian symmetric domain (or, equivalently, $\mathbf{K}$ is a maximal compact subgroup), namely

- weight 1 Hodge structures (abelian variety type): $\mathbf{D}$ is the Siegel upper half-space $\mathfrak{H}_g = \{A \in M_{g\times g}(\mathbb{C}) \mid A = A^t, \mathrm{Im}(A) > 0\} \cong \mathrm{Sp}(2g)/U(g)$; and
- weight 2 Hodge structures with $h^{2,0} = 1$ (K3 type): $\mathbf{D}$ is a Type IV domain $\{\omega \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid \omega\cdot\omega = 0,\ \omega\cdot\bar{\omega} > 0\} \cong \mathrm{SO}(2,n)/\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(n))$, where Λ is a lattice of signature $(2,k)$ (here, $\check{\mathbf{D}}$ is a quadric hypersurface in $\mathbb{P}^{k+1}$; thus $\dim\mathbf{D} = \dim\check{\mathbf{D}} = k$).

Example 1.2.1. For Hodge structures of Calabi–Yau threefold type with Hodge numbers $(1,h,h,1)$, the period domain is

$$\mathbf{D} = \mathrm{Sp}(2(1+h))/\mathrm{U}(1)\times\mathrm{U}(h).$$

The maximal compact subgroup is $\mathrm{U}(1+h)$ and the inclusion $\mathrm{U}(1)\times\mathrm{U}(h) \subset \mathrm{U}(1+h)$ induces a natural map $\mathbf{D} \to \mathfrak{H}_{1+h}$ which is neither holomorphic nor anti-holomorphic (only real analytic).

A *Variation of Hodge Structure* over S is a triple $(\mathscr{V}, F^\bullet, \nabla)$ consisting of a flat vector bundle $(\mathscr{V}, \nabla)$ together with holomorphic subbundles $F^n \subset F^{n-1} \subset \cdots \subset F^0 = \mathscr{V}$. We assume that the VHS is polarized (i.e., there is a compatible bilinear form such that (HR1) and (HR2) are satisfied). By passing to a trivialization of the associated local system we obtain a period map

$$\widetilde{P}\colon \widetilde{S} \longrightarrow \mathbf{D},$$