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Algebraic and Complex Geometry

Springer

Springer Proceedings in Mathematics & Statistics

Volume 71

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Editors

Algebraic and Complex Geometry

In Honour of Klaus Hulek's 60th Birthday

Springer

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ISSN 2194-1009 ISSN 2194-1017 (electronic)
ISBN 978-3-319-05403-2 ISBN 978-3-319-05404-9 (eBook)
DOI 10.1007/978-3-319-05404-9
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014950053

Mathematics Subject Classification (2010): 14J10, 14E30, 14J28, 14J32, 14J50, 14T05

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This volume of PROMS grew out of the international conference on "Algebraic and Complex Geometry", which took place at Leibniz Universität Hannover during the week of September 10 through 14, 2012. The event was organised on the occasion of Klaus Hulek's 60th birthday; it is with great pleasure that we dedicate this volume to him.

We would like to thank the members of the Scientific Advisory Board of the conference, David Eisenbud, Nigel Hitchin and Thomas Peternell, for their crucial input in setting up the program. The conference would have been impossible without the generous support from the following institutions:

- Deutsche Forschungsgemeinschaft
- MWK Niedersachsen
- Foundation Compositio
- GRK 1463 "Analysis, Geometry and String Theory"
- Many other institutions at Leibniz Universität Hannover

Especially, we would like to express our gratitude to all members of the Institute of Algebraic Geometry at LUH for all the helping hands before and during the conference, in particular to our secretaries Nicole Rottländer and Simone Reimann.

Our thanks go to all the authors contributing to this volume, and to the referees for the thorough job they have done. Matthias Freise took care of the compilation of this volume, and we thank him very much.

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Contents

Introduction

The following paragraphs are meant to give a brief outline of the topics of the conference "Algebraic and Complex Geometry" and the content of this PROMS volume. Algebraic and complex geometry are exceptionally active areas of research in pure mathematics which have seen many novel developments in recent years, influencing numerous other areas such as differential geometry, number theory, representation theory, and mathematical physics. Many of these interesting aspects will be reflected in what follows.

1 Topic of the Conference

The program of the conference was designed to review the latest achievements and innovations in algebraic and complex geometry. The program featured 23 lectures from various subfields, allowing a broad scope, but putting specific emphasis on two subjects of spectacular recent and ongoing progress: *geometry of moduli spaces* and *irreducible symplectic manifolds* (Hyperkähler manifolds).

Geometry of Moduli Spaces

Moduli spaces are a key object of study in algebraic and complex geometry. Originally introduced by Riemann in the case of curves, moduli spaces turned out to be interesting both for their own sake and for the numerous implications to other fields such as e.g. number theory (arithmetic geometry) and mathematical physics (string theory).

Recently, there has been a particular interest in establishing the geometric and topological properties of moduli spaces. In particular, newly developed techniques yield results on the Kodaira dimension and on the cohomology of several moduli spaces. Most of the recent results are for moduli spaces of curves, of abelian varieties and of K3 surfaces.

K3 surfaces are a special case of holomorphic symplectic manifolds, which brings us to the second central topic of the conference.

Irreducible Symplectic Manifolds (Hyperkähler Manifolds)

Irreducible symplectic manifolds (or Hyperkähler manifolds, defined by the existence of an everywhere non-degenerate holomorphic 2-form) behave in many ways similar to abelian varieties and K3 surfaces. Yet they remain quite mysterious objects.

As an illustration, there are only a few known constructions of irreducible symplectic manifolds due to Beauville, Huybrechts, Beauville-Donagi, Debarre-Voisin, and O'Grady. It is still unclear whether there might be any more. The moduli spaces of irreducible symplectic manifolds are conjectured to be locally symmetric varieties, as in the case of K3 surfaces.

The conference program highlighted several important aspects of moduli spaces and irreducible holomorphic symplectic manifolds.

For the reader's information, we decided to include a full list of speakers with titles and abstracts in the Appendix "Complete List of Talks".

At the same time, this volume reflects the broad diversity of lectures at the conference beyond the above focal topics. We continue with a short tour of the content of this book.

2 Tour of Content of This Volume

This volume comprises 11 papers on current research from different areas of algebraic and complex geometry. Reflecting the diversity of topics at the conference, we sorted the article in alphabetic order by the first author instead of grouping them by topic. Below we give a brief survey of the content.

The general topic of the paper by Barja and Stoppino concerns the relation between stability conditions and positivity in algebraic geometry. Given a 1-parameter family of polarized varieties, the authors study three different methods, all of them involving stability conditions, to prove the positivity of a natural numerical invariant associated to the family.

Beauville studies a problem related with the algebraic topology of algebraic varieties. The author expresses the second quotient of the lower central series of the fundamental group of a topological space X in terms of the homology and cohomology of X. As an application, the author considers the Fano surface parametrizing lines in a cubic threefold, where he recovers a result due to Collino.

Blume's contribution extends the classical McKay correspondence for finite subgroups G of $\mathrm{SL}(2,\mathbb{C})$ to non-algebraically closed fields. More precisely, Blume constructs for arbitrary fields K of characteristic zero a bijection between isomorphism classes of nontrivial irreducible representations of $G \subset \mathrm{SL}(2, K)$ and the irreducible components of the exceptional divisor in the minimal resolution of the quotient singularity $\mathbb{A}^2_K/G$.

The paper by Caporaso studies the interplay between the theory of linear series on algebraic curves and on graphs. To this end, the author introduces the notion of d-gonality for weighted graphs using harmonic indexed morphisms. Then a combinatorial locus of the moduli space of curves contains a d-gonal curve if the corresponding graph is d-gonal and of Hurwitz type. Conversely the dual graph of a d-gonal stable curve is equivalent to a d-gonal graph of Hurwitz type. A detailed study of the hyperelliptic case is included.

A classical problem is the subject of Catanese's considerations. He proves for a plane curve C that the map from C to its caustic is a birational map and he concludes with similar results for matrix projections.

The starting point for the extensive work of Ciliberto and Dedieu are degenerations of complex K3 surfaces. Given a degeneration of complex K3 surfaces, they investigate the limits of the corresponding Severi varieties parametrizing irreducible δ-nodal plane sections of the K3 surfaces. Applications include counting plane nodal curves through base points in special position, the irreducibility of Severi varieties of a general quartic surface, and the monodromy of the universal family of rational curves on quartic K3 surfaces.

Fujino and Gongyo consider the behaviour of divisors under smooth morphisms between smooth complex projective varieties with a special view towards nefness. Their arguments lead to a Hodge theoretic proof of the fact that nefness of the anti-canonical divisor of the source space implies the same for the target space. Previous proof of these results had been derived using positive characteristic arguments. The present work relies on a generalization of Viehweg's weak positivity theorem due to Campana.

Haydys introduced the notion of the hyperholomorphic line bundle on a hyper-kähler manifold with an S^1-action of a certain type. Previous descriptions involved twistor spaces and gauge theory, illustrating the relevance for physics. The paper by Hitchin gives examples and more general results with a more geometrical flavour.

Hollborn and Müller-Stach start from a local system $\mathbb{V}$ induced by a family of Calabi-Yau threefolds over a smooth quasi-projective curve S. Using Higgs cohomology, they determine the Hodge numbers of the cohomology group $H^1_{L^2}(S, \mathbb{V}) = H^1(\bar{S}, j_*\mathbb{V})$. This generalizes previous work to the case of quasi-unipotent local monodromies at infinity and has applications to Rohde's families of Calabi-Yau 3-folds without maximally unipotent degenerations.

Compact Kähler holomorphic-symplectic manifolds, which are deformation equivalent to the Hilbert scheme of length n subschemes of a K3 surface, are the subject of Markman's contribution. Motivated by the K3 case, Markman investigates criteria when the linear system associated with a nef line-bundle is base point free and when this linear system induces a Lagrangian fibration.

The concluding paper by Peternell and Schrack studies complex compact Kähler manifolds X carrying a contact structure (which is in some sense the opposite of a foliation). If X is almost homogeneous and $b_2(X) \geq 2$, then they show that X is a projectivised tangent bundle. Moreover, any global projective deformation of the projectivised tangent bundle over a projective space is again of this type unless it is the projectivisation of a special unstable bundle over a projective space.

Stability Conditions and Positivity of Invariants of Fibrations

M.A. Barja and L. Stoppino

Abstract We study three methods that prove the positivity of a natural numerical invariant associated to 1-parameter families of polarized varieties. All these methods involve different stability conditions. In dimension 2 we prove that there is a natural connection between them, related to a yet another stability condition, the linear stability. Finally we make some speculations and prove new results in higher dimension.

1 Introduction

The general topic of this paper regards how stability conditions in algebraic geometry imply positivity. One of the first results in this direction is due to Hartshorne [25]: a μ-semistable vector bundle of positive degree over a curve is ample. Other seminal results are Bogomolov Instability Theorem [15] and Miyaoka's Theorem on the nef cone of projective bundles over a curve [37]. These theorems – not accidentally – are recalled and used in this paper (Theorems 8 and 4).

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A. Frühbis-Krüger et al. (eds.), *Algebraic and Complex Geometry*, Springer Proceedings in Mathematics & Statistics 71, DOI 10.1007/978-3-319-05404-9_1,

An important example of this kind of result is provided by the various proofs of the so-called *slope inequality* for a non-locally trivial relatively minimal fibred surface $f: S \longrightarrow B$, with general fibre F of genus $g \geq 2$:

$$K_f^2 \geq 4\frac{g-1}{g}\chi_f.$$

There are at least three different proofs of this result. One is due to Cornalba and Harris for the Deligne-Mumford non-hyperelliptic stable case [18] (generalized to the general case by the second author [50]), and uses the Hilbert stability of the canonical morphism of the general fibre of f. In [16] Bost proves a similar result assuming Chow stability. Although the proofs of Cornalba-Harris and Bost are different, the results are almost identical, being Chow and Hilbert stability very close (Remark 15). Another proof of the slope inequality, due to Xiao [52], uses the Clifford Theorem on the canonical system of the general fibre combined with the Harder-Narashiman filtration of the vector bundle $f_*\omega_f$. A third approach has been introduced more recently by Moriwaki in [39]; this method uses the μ-stability of the kernel of the relative evaluation map $f^*f_*\omega_f \longrightarrow \omega_f$ restricted on the general fibres. In [3] there is a good account of the last two proofs. Miyaoka's Theorem is a key tool in the proof of Xiao, and Bogomolov Theorem is the main ingredient of Moriwaki's approach. So we see at least two stabilities conditions involved in the proof of the slope inequality for fibred surfaces: Hilbert (or Chow) stability and μ-stability.

In this paper we study these three methods in a general setting. Firstly we present them with arbitrary line bundles – instead of the relative canonical one – and in arbitrary dimension, when possible. Then we make a comparison between them, finding that in dimension 2 there is a yet another stability condition, the *linear stability*, that connects them. Finally we make some speculations about the higher dimensional case, and we prove a couple of new applications.

Let us describe in more detail the contents of the paper. We consider the following setting. Let $f: X \longrightarrow B$ a fibred variety, $\mathscr{L}$ a line bundle on X, and let $\mathscr{G} \subseteq f_*\mathscr{L}$ be a subsheaf of rank r. A great deal of the results presented in the paper are in a more general setting, but let us assume here for the sake of simplicity that the general fibre of $\mathscr{G}$ is generating and that $\mathscr{L}$ is nef. Following [18], we consider the number $e(\mathscr{L},\mathscr{G}) = rL^n - n\deg\mathscr{G}(L_{|F})^{n-1}$, which is an invariant of the fibration (Remark 1). We introduce the following notation (Definition 3): we say that $(\mathscr{L},\mathscr{G})$ is *f-positive* when $e(\mathscr{L},\mathscr{G}) \geq 0$. In the case $n = 2$, choosing $\mathscr{L} = \omega_f$, the slope inequality is equivalent to f-positivity of $(\omega_f, f_*\omega_f)$.

The structure of the paper is the following. In Sect. 2, after giving the first definitions, we make some useful computations via the Grothendieck-Riemann-Roch Theorem (Theorem 2 and Propositions 2 and 3): the number $e(\mathscr{L},\mathscr{G})$ appears as the leading term of a polynomial expression associated to the relative Noether morphism

$$\gamma_h \colon \mathrm{Sym}^h \mathscr{G} \longrightarrow f_*\mathscr{L}^{\otimes h}, \text{ for } h \gg 0.$$

We then give a new elementary proof of a consequence of Miyaoka's result (Theorem 3): if $\mathscr{L}$ is nef and $\mathscr{G}$ is sheaf semistable, then $(\mathscr{L}, \mathscr{G})$ is f-positive. This is the first case we see where a stability condition implies f-positivity.

In Sect. 3 we describe the three methods, adding here and there some new contribution. As an illustration we re-prove along the way the slope inequality for fibred surfaces via the three methods (Examples 2, 3, and 5). The neat idea would be to extend them so that they all give as an output f-positivity of the couple $(\mathscr{L}, \mathscr{G})$, under some suitable assumptions. The Cornalba-Harris and Bost methods are originally stated in the general setting; we present them providing a slight generalization of the first one. They prove f-stability with the assumption that the fibre over general $t \in B$ is Hilbert or Chow semistable together with the morphism defined by the fibre $G_t := \mathscr{G} \otimes \mathbb{C}(t)$ (Theorems 6 and 7).

After discussing these methods, we make in Sect. 3.2 a digression on some applications that are specific to the Cornalba-Harris method. In particular we give in Proposition 4 a bound on the canonical slope of the fibred surfaces such that the k-th Hilbert point of (F, ω_F) is semistable for *fixed* k. This suggests a possible meaningful stratification of the moduli space of curves $\mathscr{M}_g$.

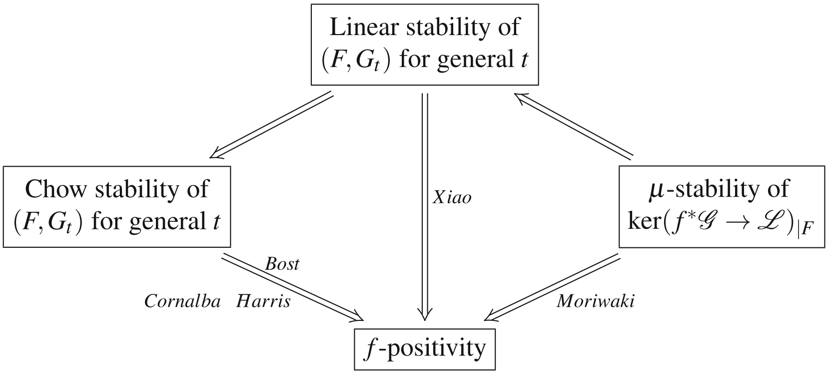
The method of Xiao was extended in higher dimensions by Konno [30] and Ohno [45]. We give a general compact version (Proposition 5). Xiao's method does not provide in general f-positivity; it gives an inequality between the invariants L^n and $\deg \mathscr{G}$ that has to be interpreted case by case.

Moriwaki's method is described in Sect. 3.4. It only works in dimension 2, and it gives f-positivity if the restriction of the kernel sheaf $\ker(f^*\mathscr{G} \longrightarrow \mathscr{L})$ is μ-semistable on the general fibres. We also provide a new condition for f-positivity, independent from the one of the theorem of Moriwaki (Theorem 10).

It is natural to try and make a comparison between these results, and between their assumptions: in particular, in the case of fibred surfaces all the three methods work because the canonical system enjoys many different properties or is there a red thread binding the three approaches? In Sect. 4 we study the 2-dimensional case. It turns out that there is a yet another stability concept, the *linear stability*, playing a central role in all three methods. Indeed, we observe the following:

- Section 4.1: linear (semi-)stability can be assumed as hypothesis in the Cornalba Harris method, as it implies Chow (semi-)stability (Mumford and others).
- Section 4.2: linear (semi-)stability is the key assumptions that assures that the method of Xiao produces f-positivity.
- Section 4.3: linear (semi-)stability is implied by the stability assumption needed in Moriwaki's method and in a large class of cases is equivalent to it (Mistretta-Stoppino).

So the picture goes as follows:



In Sect. 4.2 we also prove some positivity results that can be proved via Xiao's method with weaker assumptions.

Finally in Sect. 5 we consider the higher dimensional case. At this state of art, there is no hope to reproduce in higher dimension the beautiful connection between the three methods described for dimension 2. First of all, the method of Moriwaki seemingly can not even be extended to dimension higher than 2 (Remark 21). However, we provide some results regarding the other two methods. Firstly we prove that the hypothesis of linear stability still implies a positivity result via Xiao's method (Proposition 11). In Sect. 5.2, using known stability results, we can prove new inequalities for families of abelian varieties and of K3 surfaces via the Cornalba-Harris and Bost methods. Moreover, we conjecture a higher-dimensional slope inequality to hold for fibred varieties whose relative canonical sheaf is relatively nef and ample (Conjecture 1). We end the paper with an application of the (conjectured) slope inequality in higher dimension: using the techniques of Pardini [47] it is possible to derive from the slope inequality a sharp Severi inequality $K_X^n \geq 2n!\chi(\omega_X)$ for n-dimensional varieties with maximal Albanese dimension (Proposition 14). It is worth noticing that in [4] the first author proves this Severi inequality, and Severi type inequalities for any nef line bundle, independently of such conjectured slope inequality.

2 First Results

2.1 First Definitions and Motivation

We work over the complex field. All varieties, unless differently specified, will be normal and projective. Given a line bundle $\mathscr{L}$ on a variety X, we call L any (Cartier) divisor associated. It is possible to develop the major part of the theory for reflexive sheaves associated to Weil $\mathbb{Q}$-Cartier divisors, but in order to avoid cumbersome arguments, we will stitch to this setting.

Let X be a variety of dimension n, and B a smooth projective curve. Let $f: X \longrightarrow B$ be a flat proper morphism with connected fibres. Throughout the paper we shall call this data $f: X \longrightarrow B$ a *fibred variety*.

Let $\mathscr{L}$ be a line bundle on X. The pushforward $f_*\mathscr{L}$ is a torsion free coherent sheaf on the base B, hence it is locally free because B is smooth 1-dimensional. Let $\mathscr{G} \subseteq f_*\mathscr{L}$ be a subsheaf of rank r. The sheaf $\mathscr{G}$ defines a family of r-dimensional linear systems on the fibres of f,

$$G_t := \mathscr{G} \otimes \mathbb{C}(t) \subseteq H^0(F, \mathscr{L}_{|F}),$$

where $t \in B$ and $F = f^*(t)$. Let us recall that the evaluation morphism

$$ev: f^*\mathscr{G} \longrightarrow \mathscr{L}$$

is surjective at every point of X if and only if it induces a morphism φ from X to the relative projective bundle $\mathbb{P} := \mathbb{P}_B(\mathscr{G})$ over B

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{P}_B(\mathscr{G}) := \mathbb{P} \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \pi} & \\ B & & \end{array}$$

such that $\mathscr{L} = \varphi^*(\mathscr{O}_{\mathbb{P}}(1))$. We will denote the surjectivity condition for ev by saying that the sheaf $\mathscr{G}$ is *generating* for $\mathscr{L}$. If ev is only generically surjective, it defines a rational map $\varphi: X \dashrightarrow \mathbb{P}$. In this case, let D be the unique effective divisor such that $f^*\mathscr{G} \longrightarrow \mathscr{L}(-D)$ is surjective in codimension 1. The divisor D is called the *fixed locus* of $\mathscr{G}$ in X. Clearly the evaluation morphism $f^*\mathscr{G} \longrightarrow \mathscr{L}(-D)$ is surjective in codimension 1.

Moreover, by Hironaka's Theorem, there exist a desingularization $\nu: \tilde{X} \longrightarrow X$ and a morphism $\tilde{\varphi}: \tilde{X} \longrightarrow \mathbb{P}$ such that $\tilde{\varphi} = \varphi \circ \nu$, and an effective ν-exceptional divisor E on $\tilde{X}$ such that

$$\tilde{\varphi}^*(\mathscr{O}_{\mathbb{P}}(1)) \cong \nu^*(\mathscr{L}(-D)) \otimes \mathscr{O}_{\tilde{X}}(-E).$$

See [45, Lemma 1.1] for a detailed proof of these facts. Define $\mathscr{M} := \tilde{\varphi}^*\mathscr{O}_{\mathbb{P}}(1) \subseteq \nu^*\mathscr{L}$; following [45] we call this the *moving part* of the couple $(\mathscr{L}, \mathscr{G})$, and we define the *fixed part* of $(\mathscr{L}, \mathscr{G})$ on $\tilde{X}$ to be $Z := \nu^*(D) + E$. Call $\tilde{f} := f \circ \nu$ the induced fibration. Clearly the evaluation homomorphism $\tilde{f}^*\mathscr{G} \longrightarrow \mathscr{M}$ is surjective at every point of $\tilde{X}$, i.e. $\mathscr{G}$ is generating for $\mathscr{M}$ on $\tilde{X}$.

Example 1. Let $f: S \longrightarrow B$ be a fibred surface, assuming for simplicity that S is smooth. Let $\omega_f = \omega_S \otimes f^*\omega_B^{-1}$ be the relative dualizing sheaf of f. Let g be the (arithmetic) genus of the fibres. The general fibres are smooth curves of genus g.

Let us assume that $g \geq 2$: then the restriction of ω_f on the general fibres is ample. Hence the base divisor D is vertical with respect to f. Moreover, the line bundle ω_f has negative degree only on the $(-1)-$curves contained in the fibres. So, all the vertical $(-1)-$curves of S are contained in D. It is possible to contract these curves preserving the fibration, and obtaining a unique *relatively minimal* fibration associated whose relative dualizing sheaf is f-nef. However, there could still be a divisorial fixed locus, as we see now for the case of nodal fibrations.

Let us suppose that f is a nodal fibration, i.e. that any fibre of f is a reduced curve with only nodes as singularities. We now describe explicitly the moving and the fixed part of $(\omega_f, f_*\omega_f)$. Let us first recall the following simple result, that can be found in [39, Prop. 2.1.3]. If C is a nodal curve, the base locus of ω_C is given by all the disconnecting nodes and all the smooth rational components of C that are attached to the rest of the fibre only by disconnecting nodes; following [39] we call these components of *socket type*.

The fixed locus of $(\omega_f, f_*\omega_f)$ is the union D of all components of socket type. Indeed, by what observed above the evaluation homomorphism *ev*: $f^*f_*\omega_f \longrightarrow \omega_f$ factors through $\omega_f(-D)$. On the other hand, it is easy to verify that the restriction of $\omega_f(-D)$ on any fibre is well defined except that on the disconnecting nodes not lying on components of socket type, so $f^*f_*\omega_f(-D) \longrightarrow \omega_f(-D)$ is surjective in codimension one.

Let $\nu\colon \tilde{S} \longrightarrow S$ be the blow up of all the base points of the map induced by $f_*\omega_f(-D)$; call E the exceptional divisor, and $\tilde{f} = f \circ \nu$ the induced fibration on $\tilde{S}$. Then we have that all the components of E are of socket type for the corresponding fibre, and that the union of all the components of socket type of the fibres of $\tilde{f}$ is $\tilde{D} + E$, where $\tilde{D}$ is the inverse image of D. Thus $\tilde{D} + E$ is the fixed part of $(\omega_{\tilde{f}}, \tilde{f}_*\omega_{\tilde{f}})$, and the evaluation homomorphism

$$\tilde{f}^*\tilde{f}_*\omega_{\tilde{f}}(-\tilde{D}-E) \longrightarrow \omega_{\tilde{f}}(-\tilde{D}-E)$$

is surjective at every point. Noting that $\omega_{\tilde{f}} \cong \nu^*(\omega_f) \otimes \mathscr{O}_{\tilde{S}}(E)$ (see for instance [10, Chap. 1, Theorem 9.1]), we have that

$$\omega_{\tilde{f}}(-\tilde{D}-E) \cong \nu^*(\omega_f) \otimes \mathscr{O}_{\tilde{S}}(-\tilde{D}) \cong \nu^*(\omega_f(-D)) \otimes \mathscr{O}_{\tilde{S}}(-E).$$

So the moving part of $(\omega_f, f_*\omega_f)$ is $\mathscr{M} \cong \nu^*(\omega_f(-D)) \otimes \mathscr{O}_{\tilde{S}}(-E)$.

Let us now come to the definition of the main characters of the play.

Definition 1. With the above notation, define the *Cornalba-Harris invariant*

$$e(\mathscr{L},\mathscr{G}) := rL^n - n\deg\mathscr{G}(L_{|F})^{n-1},$$

where L is a divisor such that $\mathscr{L} \cong \mathscr{O}_X(L)$, and F is a general fibre.

Remark 1. The number $e(\mathscr{L},\mathscr{G})$ is indeed invariant by twists of line bundles from the base curve B. Indeed, if $\mathscr{A}$ is a line bundle on B we have

$$\operatorname{rank}(\mathscr{G}\otimes\mathscr{A})=\operatorname{rank}\mathscr{G}=r,\quad \deg(\mathscr{G}\otimes\mathscr{A})=\deg\mathscr{G}+r\deg\mathscr{A},$$

$$(L+f^*A)^n=L^n+n\deg\mathscr{A}\,L_{|F}^{n-1},\quad (\mathscr{L}\otimes f^*\mathscr{A})_{|F}\cong\mathscr{L}_{|F}.$$

It is therefore immediate to verify that $e(\mathscr{L}\otimes f^*\mathscr{A},\mathscr{G}\otimes\mathscr{A})=e(\mathscr{L},\mathscr{G})$.

Remark 2. There is another significant incarnation of the C-H invariant: the number $r^{n-1}e(\mathscr{L},\mathscr{G})$ is the top self-intersection of the divisor $rL-\deg\mathscr{G}F$.

Let us now consider again a fibred surface $f\colon S\longrightarrow B$ as in Example 1. Let $g\geq 2$ be the genus of the fibres and b the genus of the base curve B. The main relative invariants for f are $K_f^2=K_S^2-8(b-1)(g-1)$ and $\chi_f=\chi(\mathscr{O}_S)-\chi(\mathscr{O}_B)\chi(\mathscr{O}_F)=\chi(\mathscr{O}_S)-(g-1)(b-1)$. By Leray's spectral sequence and Riemann-Roch one sees that $\chi_f=\deg f_*\omega_f$. The *canonical slope* s_f of the fibration is defined as the ratio between K_f^2 and χ_f. The slope s_f have been extensively studied in the literature (see [3, 6, 52]).

In a more general setting, given a line bundle $\mathscr{L}$ on X and a subsheaf $\mathscr{G}\subseteq f_*\mathscr{L}$, one can consider, when possible, the ratio between L^n and $\deg\mathscr{G}$, as follows.

Definition 2. With the same notation as above, let us suppose moreover that $\deg\mathscr{G}>0$. We define the *slope of the couple* $(\mathscr{L},\mathscr{G})$ as

$$s_f(\mathscr{L},\mathscr{G}):=\frac{L^n}{\deg\mathscr{G}}.$$

When $\mathscr{G}=f_*\mathscr{L}$, we shall use the notation $s_f(\mathscr{L})$.

There is a rich literature about the search of lower bounds for the slope, in particular about the canonical one. The most general result is the following (see [5]).

Proposition 1. *Assume that $\mathscr{L}$ and $f_*\mathscr{L}$ are nef. Then $s_f(\mathscr{L})\geq 1$.*

This bound is attained by a projective bundle on B and its tautological line bundle.

Remark 3. The slope is *not* invariant by twists of line bundles. Indeed, let $F=f^*(t)$ be a general fibre, and $G_t:=\mathscr{G}\otimes\mathbb{C}(t)\subseteq H^0(F,\mathscr{L}_{|F})$. Attached to the triple $(f,\mathscr{G},\mathscr{L})$ a *natural* ratio appears, which depends on the geometry of the triple $(F,G_t,\mathscr{L}_{|F})$. Indeed, consider the line bundle $\mathscr{L}(kF)$ obtained by "perturbing" $\mathscr{L}$ with kF for $k\in\mathbb{N}$, and the corresponding perturbed sheaf $\mathscr{G}\otimes\mathscr{O}_B(kt)\subseteq f_*(\mathscr{L}(kF))\cong f_*\mathscr{L}\otimes\mathscr{O}_B(kt)$. Then we have that

$$s_f(\mathscr{L},\mathscr{G})(k):=s_f(\mathscr{L}(kF),\mathscr{G}\otimes\mathscr{O}_B(kt))=\frac{(L+kF)^n}{\deg\mathscr{G}\otimes\mathscr{O}_B(kt)}=\frac{L^n+kn(L_{|F})^{n-1}}{\deg(\mathscr{G})+k\operatorname{rank}\mathscr{G}}.$$

Hence

$$\lim_{k\to\infty}s_f(\mathscr{L},\mathscr{G})(k)=n\frac{(L_{|F})^{n-1}}{\operatorname{rank}\mathscr{G}}.$$