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Positivity in Algebraic Geometry, II

Positivity for Vector Bundles, and Multiplier Ideals

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To Lee Yen, Sarah and John

Preface

The object of this book is to give a contemporary account of a body of work in complex algebraic geometry loosely centered around the theme of positivity.

Our focus lies on a number of questions that grew up with the field during the period 1950–1975. The sheaf–theoretic methods which revolutionized algebraic geometry in the fifties — notably the seminal work of Kodaira, Serre and Grothendieck — brought into relief the special importance of ample divisors. By the mid sixties a very satisfying theory of positivity for line bundles was largely complete, and first steps were taken to extend the picture to bundles of higher rank. In a related direction, work of Zariski and others led to a greatly deepened understanding of the behavior of linear series on algebraic varieties. At the border with topology the classical theorems of Lefschetz were understood from new points of view, and extended in surprising ways. Hartshorne's book [252] and the survey articles in the Arcata proceedings [257] give a good picture of the state of affairs as of the mid seventies.

The years since then have seen continued interest and activity in these matters. Work initiated during the earlier period has matured and found new applications. More importantly, the flowering of higher dimensional geometry has led to fresh perspectives and — especially in connection with vanishing theorems — vast improvements in technology. It seems fair to say that the current understanding of phenomena surrounding positivity goes fundamentally beyond what it was thirty years ago. However many of these new ideas have remained scattered in the literature, and others up to now have not been worked out in a systematic fashion. The time seemed ripe to pull together some of these developments, and the present volumes represent an attempt to do so.

The book is divided into three parts. The first, which occupies Volume I, focuses on line bundles and linear series. In the second volume, Part Two part takes up positivity for vector bundles of higher ranks. Part Three deals with ideas and methods coming from higher dimensional geometry, in the form of multiplier ideals. A brief introduction appears at the beginning of each of the parts.

I have attempted to aim the presentation at non-specialists. Not conceiving of this work as a text, I haven't started from a clearly defined set of prerequisites. But the subject is relatively non-technical in nature, and familiarity with the canonical texts [256] and [224] (combined with occasional faith and effort) is more than sufficient for the bulk of the material. In places — for example, Chapter 4 on vanishing theorems — our exposition is if anything more elementary than the standard presentations.

I expect that many readers will want to access this material in short segments rather than sequentially, and I have tried to make the presentation as friendly as possible for browsing. At least a third of the book is devoted to concrete examples, applications and pointers to further developments. The more substantial of these are often collected together into separate sections. Others appear as Examples or Remarks (typically distinguished by the presence and absence respectively of indications of proof). Sources and attributions are generally indicated in the body of the text: these references are supplemented by brief sections of Notes at the end of each chapter.

We work throughout with algebraic varieties defined over the complex numbers. Since substantial parts of the book involve applications of vanishing theorems, hypotheses of characteristic zero are often essential. However I have attempted to flag those foundational discussions that extend with only minor changes to varieties defined over algebraically closed fields of arbitrary characteristic. By the same token we often make assumptions of projectivity when in reality properness would do. Again I try to provide hints or references for the more general facts.

Although we use the Hodge decomposition and the Hard Lefschetz theorem on several occasions, we say almost nothing about the Hodge-theoretic consequences of positivity. Happily these are treated in several sources, most recently in the beautiful book [553], [552] of Voisin. Similarly, the reader will find relatively little here about the complex analytic side of the story. For this we refer to Demailly's notes [110] or his forthcoming book [104].

Concerning matters of organization, each chapter is divided into several sections, many of which are further partitioned into subsections. A reference to Section 3.1 points to the first section of Chapter 3, while Section 3.1.B refers to the second subsection therein. Statements are numbered consecutively within each section: so for example Theorem 3.1.17 indicates a result from Section 3.1 (which, as it happens, appears in 3.1.B). As an aid to the reader, each of the two volumes reproduces the table of contents of the other. The index and list of references cover both volumes. Large parts of Volume I can be read without access to Volume II, but Volume II makes frequent reference to Volume I.

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This project has profited from collaborations with a number of co-authors, including Jean-Pierre Demailly, Mark Green and Karen E. Smith. Joint work many years ago with Bill Fulton has helped to shape the content and presentation of Chapter 3 and Part Two. I likewise owe a large mathematical debt to Lawrence Ein: I either learned from him or worked out together with him a significant amount of the material in Part Three, and our collaboration has had an influence in many other places as well.

I am grateful to several individuals that made particularly valuable contributions to the preparation of these volumes. János Kollár convinced me to start the book, and Bill Fulton insisted (on several occasions) that I finish it. Besides their encouragement, they contributed detailed suggestions from careful readings of drafts of several chapters. I also received copious comments on different parts of a preliminary draft from Jun-Muk Hwang, Steve Kleiman and Karen Smith. Olivier Debarre read through the draft in its entirety, and provided a vast number of corrections and improvements.

Like many first-time authors, I couldn't have imagined when I began writing how long and consuming this undertaking would become. I'd like to take this opportunity to express my profound appreciation to the colleagues, friends and family members that offered support, encouragement and patience along the way.

December, 2003 Robert Lazarsfeld

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Part Three: Multiplier Ideals and Their Applications

Contents of Volume I

Notation and Conventions

For the most part we follow generally accepted notation, as in [256]. We do however adopt a few specific conventions:

- We work throughout over the complex numbers $\mathbf{C}$.
- A *scheme* is an algebraic scheme of finite type over $\mathbf{C}$. A *variety* is a reduced and irreducible scheme. We deal exclusively with closed points of schemes.
- If X is a variety, a *modification* of X is a projective birational mapping $\mu : X' \longrightarrow X$.
- Given an irreducible variety X, we say that a property holds at a *general* point of X if it holds for all points in the complement of a proper algebraic subset. A property holds at a *very general* point if it is satisfied off the union of countably many proper subvarieties.
- Let $f : X \longrightarrow Y$ be a morphism of varieties or schemes, and $\mathfrak{a} \subseteq \mathcal{O}_Y$ an ideal sheaf on Y. We sometimes denote by $f^{-1}\mathfrak{a} \subseteq \mathcal{O}_X$ the ideal sheaf $\mathfrak{a} \cdot \mathcal{O}_X$ on X determined by $\mathfrak{a}$. While strictly speaking this conflicts with the notation for the sheaf-theoretic pullback of $\mathfrak{a}$, we trust that no confusion will result.
- If E is a vector bundle on a scheme X, $\mathbf{P}(E)$ denotes the projective bundle of one-dimensional *quotients* of E. On a few occasions we will want to work with the bundle of one-dimensional *subspaces* of E: this is denoted $\mathbf{P}_{\text{sub}}(E)$. Thus
$$\mathbf{P}_{\text{sub}}(E) \ = \ \mathbf{P}(E^*).$$
A brief review of basic facts concerning projective bundles appears in Appendix **??**.
- Given a vector space or vector bundle E, $S^k E$ denotes the k^{th} symmetric power of E, and $\text{Sym}(E) = \oplus S^k(E)$ is the full symmetric algebra on E.
- Let X_1, X_2 be varieties or schemes. Without specific comment we write

$$\mathrm{pr}_1 : X_1 \times X_2 \longrightarrow X_1 \quad , \quad \mathrm{pr}_2 : X_1 \times X_2 \longrightarrow X_2$$

for the two projections of $X_1 \times X_2$ onto its factors. When other notation for the projections seems preferable, we introduce it explicitly.

- Given a real-valued function $f : \mathbf{N} \longrightarrow \mathbf{R}$ defined on the natural numbers, we say that $f(m) = O(m^k)$ if

$$\limsup_{m \to \infty} \frac{|f(m)|}{m^k} \ < \ \infty.$$

Part Two

Positivity for Vector Bundles