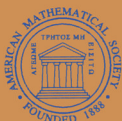


Mathematical
Surveys
and
Monographs

Volume 9

Linear Approximation

Arthur Sard



American Mathematical Society

Other Titles in This Series

- 45 **George M. Bergman and Adam O. Hausknecht**, *Cogroups and co-rings in categories of associative rings*, 1996
- 44 **J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo**, *Fundamental groups of compact Kähler manifolds*, 1996
- 43 **James E. Humphreys**, *Conjugacy classes in semisimple algebraic groups*, 1995
- 42 **Ralph Freese, Jaroslav Ježek, and J. B. Nation**, *Free lattices*, 1995
- 41 **Hal L. Smith**, *Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems*, 1995
- 40.2 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon**, *The classification of the finite simple groups*, number 2, 1995
- 40.1 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon**, *The classification of the finite simple groups*, number 1, 1994
- 39 **Sigurdur Helgason**, *Geometric analysis on symmetric spaces*, 1993
- 38 **Guy David and Stephen Semmes**, *Analysis of and on uniformly rectifiable sets*, 1993
- 37 **Leonard Lewin, Editor**, *Structural properties of polylogarithms*, 1991
- 36 **John B. Conway**, *The theory of subnormal operators*, 1991
- 35 **Shreeram S. Abhyankar**, *Algebraic geometry for scientists and engineers*, 1990
- 34 **Victor Isakov**, *Inverse source problems*, 1990
- 33 **Vladimir G. Berkovich**, *Spectral theory and analytic geometry over non-Archimedean fields*, 1990
- 32 **Howard Jacobowitz**, *An introduction to CR structures*, 1990
- 31 **Paul J. Sally, Jr. and David A. Vogan, Jr., Editors**, *Representation theory and harmonic analysis on semisimple Lie groups*, 1989
- 30 **Thomas W. Cusick and Mary E. Flahive**, *The Markoff and Lagrange spectra*, 1989
- 29 **Alan L. T. Paterson**, *Amenability*, 1988
- 28 **Richard Beals, Percy Deift, and Carlos Tomei**, *Direct and inverse scattering on the line*, 1988
- 27 **Nathan J. Fine**, *Basic hypergeometric series and applications*, 1988
- 26 **Hari Bercovici**, *Operator theory and arithmetic in H^∞* , 1988
- 25 **Jack K. Hale**, *Asymptotic behavior of dissipative systems*, 1988
- 24 **Lance W. Small, Editor**, *Noetherian rings and their applications*, 1987
- 23 **E. H. Rothe**, *Introduction to various aspects of degree theory in Banach spaces*, 1986
- 22 **Michael E. Taylor**, *Noncommutative harmonic analysis*, 1986
- 21 **Albert Baernstein, David Drasin, Peter Duren, and Albert Marden, Editors**, *The Bieberbach conjecture: Proceedings of the symposium on the occasion of the proof*, 1986
- 20 **Kenneth R. Goodearl**, *Partially ordered abelian groups with interpolation*, 1986
- 19 **Gregory V. Chudnovsky**, *Contributions to the theory of transcendental numbers*, 1984
- 18 **Frank B. Knight**, *Essentials of Brownian motion and diffusion*, 1981
- 17 **Le Baron O. Ferguson**, *Approximation by polynomials with integral coefficients*, 1980
- 16 **O. Timothy O'Meara**, *Symplectic groups*, 1978
- 15 **J. Diestel and J. J. Uhl, Jr.**, *Vector measures*, 1977
- 14 **V. Guillemin and S. Sternberg**, *Geometric asymptotics*, 1977
- 13 **C. Pearcy, Editor**, *Topics in operator theory*, 1974

This page intentionally left blank

MATHEMATICAL Surveys and Monographs

Volume 9

Linear Approximation

Arthur Sard



American Mathematical Society
Providence, Rhode Island

The writing of this book was supported in part by the Aeronautical Research Laboratories and the Office of Scientific Research, both of the United States Air Force.

Library of Congress Catalog Number: 63-11988
International Standard Book Number 0-8218-1509-1
International Standard Serial Number 0076-5376

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.


Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to reprint-permission@ams.org.

© Copyright 1963 by the American Mathematical Society. All rights reserved.
Printed in the United States of America.

Reprinted with corrections, 1982

The American Mathematical Society retains all rights
except those granted to the United States Government.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

 Printed on recycled paper.

10 9 8 7 6 5 4 01 00 99 98 97 96

Preface

The form in which I have written "Linear approximation" seems to me to be suited to the subject, and to the use of mathematicians, scientists, and engineers.

Readers interested in the applications may wish to start with the illustrative examples and the statements of theorems. All readers are urged to skip boldly and to sample where they will.

I am grateful to my wife and to many men and women for help and teaching. In the words of David, Psalm 16, 6,

חבליים נפלו לי בנעימים אף נחלת שפרה עליי.

*The lines are fallen unto me in pleasant places;
yea, I have a goodly heritage.*

ARTHUR SARD

July 21, 1962

QUEENS COLLEGE
THE CITY UNIVERSITY OF NEW YORK
FLUSHING, NEW YORK

Second Preface

There has been much done since 1963. Rather than attempting a synopsis, I will list a number of my papers, in which detailed references are given to work of Atteia, Barnhill, DeBoor and Lynch, Delves, M. Golomb and Weinberger, Holladay, Schempp, Schoenberg, Walsh, Ahlberg and Nilson, myself, and others.

The “special formulas” of Chapter 2 are now called “natural splines” or “minimizing splines”. The open question discussed on page 107 has been resolved affirmatively: The natural splines are the best broad formulas. See references [3] and [9] of the Supplementary Bibliography. Furthermore, the best formulas of Parts 1, 3, and 4 of Chapter 2 are derivable from Part 2, since the best approximation of Gx is Gy , where y is the natural spline approximation of x and G is any admissible operator. The direct calculation of Gy may, however, be easier than that of y .

Papers [1] and [7] refer to Chapter 6; paper [6] refers to Chapter 9.

I wish to thank Valerie Meyer for her great help.

ARTHUR SARD

August 15, 1980

BINNINGEN
SWITZERLAND

Table of Contents

INTRODUCTION	1
Functionals	1
General linear formulas	5
The effect of error in input	7
The use of probability	8
Efficient approximation	9
Minimal response to error. Variance	10
Other topics	10
CHAPTER 1. FUNCTIONALS IN TERMS OF DERIVATIVES	11
The spaces C_n , C_n , V of functions	11
The space \mathcal{C}_n^* of functionals	13
A standard form for elements of \mathcal{C}_{n-1}^*	14
Figure 1. Step functions and their integrals	16
Inequalities	19
Symmetry and skew symmetry	23
Functionals that vanish for degree $n-1$	25
An approximation of $\int_{-1}^1 x(s) ds$	26
An approximation of $\int_0^1 x(s) ds/\sqrt{s}$	31
An approximation of the derivative $x_1(s)$ at $s = 1/4$	31
Linear interpolation	32
A theorem on convex families of functions	33
CHAPTER 2. APPLICATIONS	36
Part 1. Integrals	36
Introduction. Best formulas	36
The approximation of $\int_0^m x$ by $c_0x(0) + \dots + c_mx(m)$	51
Best and nearly best integration formulas	53
Derivation of the formulas	61
m even	65
m odd	71
A formula of Gaussian type involving two ordinates	74
Approximations of Gaussian type	80
Approximations that involve derivatives	81
Stepwise solution of differential equations	82

Part 2. Values of functions	84
Conventional interpolation with distinct arguments	84
Conventional interpolation with at most q coincident arguments	88
Interpolation	90
Special formulas and best narrow formulas	93
Broad interpolation	104
Interpolations that use derivatives	107
Part 3. Derivatives	111
Conventional approximate differentiation	111
Best formulas	111
Part 4. Sums	116
An instance	116
CHAPTER 3. LINEAR CONTINUOUS FUNCTIONALS ON C_n	118
Normed linear spaces	118
Additive operators	122
The adjoint space	125
Riesz's Theorem	127
The space C_n^*	138
An illustration	144
The spaces Z and Z^*	145
Taylor operators	151
The operator δ	154
The spaces B_n and K_n	155
CHAPTER 4. FUNCTIONALS IN TERMS OF PARTIAL DERIVATIVES	160
The space $B_{p,q}$	160
Taylor's formula	162
The spaces $\mathcal{K}_{p,q}^*$ and $B_{p,q}$	170
The spaces B	181
The spaces \mathcal{K}^* and B	194
Symmetry	202
Appraisals	203
Figures and tables	206
CHAPTER 5. APPLICATIONS	214
Approximation of an integral in terms of its integrand at the center of mass	214
Circular domain of integration	221
Use of several values of the integrand	224
Use of derivatives of the integrand	226
A functional not in \mathcal{K}^* unless n is large	229
Circular domain and partial derivatives	230
An interpolation	232
Double linear interpolation	232
An approximate differentiation	233

CHAPTER 6. LINEAR CONTINUOUS FUNCTIONALS ON B, Z, K	240
The norm in B	240
Riesz's Theorem	242
The space B^*	246
The space \mathcal{B}^*	249
Illustration	252
The operators δ_s and δ_t	256
The spaces Z and Z^*	256
The space K	262
The norm in K	265
The space K^*	266
K^* as a subspace of B^*	269
CHAPTER 7. FUNCTIONS OF m VARIABLES	271
The space B	271
The full core ϕ	274
The norm in B	278
Functions of bounded variation	279
The space B^*	279
The space \mathcal{B}^*	280
The spaces Z and Z^*	281
The covered core	281
The retracted core ρ and the space K	284
The norm in K	286
The space K^*	287
The space \mathcal{K}^*	288
K^* as a subspace of B^*	289
Illustration	290
Figures and tables	295
CHAPTER 8. FACTORS OF OPERATORS	300
Banach spaces	301
Baire's Theorem	303
The inverse of a linear continuous map	305
The factor space X/X_0	308
The quotient theorem	310
Import thereof	313
An instance in which $U = D_s^n$	314
Related instances	315
U a linear homogeneous differential operator	316
Approximation of a function by a solution of a linear homogeneous differential equation	316
A trigonometric approximation	322
An instance in which U involves difference operators	324