A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods

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Preface to the Classic Edition

The validity of the first paragraph of the Preface to the original edition

> Applications of the theory of Lie groups, Lie algebras and their representations are many and varied. This is a rapidly growing field through which one can bring to bear many powerful methods of modern mathematics.

written 18 years ago, has been borne out by the profusion of published work in this area.

During this period of time we have seen the appearance of several other presentations of Lie theory directed to various special fields. Another trend has been the increased emphasis on computational Lie algebra, with heavy reliance on computers.

We have naturally been gratified and much encouraged by the warm reception of the first edition. When it became known that a classic edition was to be published, we received many valuable suggestions for additional topics to be included. We regret that most of these could not be readily incorporated without completely changing the outlook, and greatly increasing the size, of the book. We have corrected a few minor misprints which appeared in the original edition, but the character of the book, especially its focus on the classical representation theory and its computational aspects, has not changed.

We thank the many people who wrote us with their comments and suggestions. We also thank the entire staff of SIAM for their interest and unfailing cooperation during all phases of this project.

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We are grateful to Miss Susan R. Gershuni who typed the entire manuscript and its revisions and to our wives, Irene and Lillie, who helped with the preparation of the manuscript.

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INTRODUCTION

The theory of Lie groups and Lie algebras is an area of mathematics in which we can see a harmonious interaction between the methods of classical analysis and modern algebra. This theory, a direct outgrowth of a central problem in the calculus, has today become a synthesis of many separate disciplines, each of which has left its own mark.

The theory of Lie groups was developed by the Norwegian mathematician Sophus Lie in the late nineteenth century in connection with his work on systems of differential equations. Lie groups arise in the study of solutions of differential equations just as finite groups arise in the study of algebraic equations. The possibility of solving many differential equations which arise in practical problems can often be traced to some geometrical or other symmetry property of the problem. Indeed, Lie groups play a fundamental role in many parts of geometry itself, and have contributed significantly to the development of that subject. Thus, for many years Lie theory has been found useful in such areas of pure mathematics as the differential geometry of symmetric spaces. Conversely, geometrical methods have played a dominant role in the structure theory of Lie groups and Lie algebras.

In our exposition of this structure theory, we focus on the classification of real and complex semisimple Lie algebras. We found this to be an excellent place to introduce and apply many of the tools of classical linear algebra and vector space theory, as well as the modern theory of modules.

The importance of Lie algebras and Lie groups for applied mathematics and for physics has also become increasingly evident in recent years. In applied mathematics, Lie theory remains a powerful tool for studying differential equations, special functions and perturbation theory. Lie theory often enters into physics either through the presence of exact kinematical symmetries or through the use of idealized dynamical models having greater symmetry than is present in the real world. These exact kinematical symmetries include rotational, translational and Galilean or Lorentz invariance, as well as symmetries arising from the use of canonical formalism in both classical and quantum mechanics. Broken symmetries arising from approximate models are encountered in atomic and nuclear spectroscopy, and in elementary particle physics.