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(continued after index)

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Linear Algebra Through Geometry

Second Edition

With 92 Illustrations

Springer

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Library of Congress Cataloging-in-Publication Data
Banchoff, Thomas.
Linear algebra through geometry, 2nd ed./Thomas Banchoff, John Wermer.
p. cm.—(Undergraduate texts in mathematics)
Includes bibliographical references and index.
ISBN-13: 978-1-4612-8752-0 e-ISBN-13: 978-1-4612-4390-8
DOI 10.1007/978-1-4612-4390-8
1. Algebras, Linear. I. Wermer, John. II. Title. III. Series.
QA184.B36 1991
512'.5—dc20 91-18083

Printed on acid-free paper.

Typeset by Asco Trade Typesetting Ltd., Hong Kong.

9 8 7 6 5 4 3

To our wives Lynore and Kerstin

Preface to the Second Edition

In this book we lead the student to an understanding of elementary linear algebra by emphasizing the geometric significance of the subject.

Our experience in teaching undergraduates over the years has convinced us that students learn the new ideas of linear algebra best when these ideas are grounded in the familiar geometry of two and three dimensions. Many important notions of linear algebra already occur in these dimensions in a non-trivial way, and a student with a confident grasp of the ideas will encounter little difficulty in extending them to higher dimensions and to more abstract algebraic systems. Moreover, we feel that this geometric approach provides a solid basis for the linear algebra needed in engineering, physics, biology, and chemistry, as well as in economics and statistics.

The great advantage of beginning with a thorough study of the linear algebra of the plane is that students are introduced quickly to the most important new concepts while they are still on the familiar ground of two-dimensional geometry. In short order, the student sees and uses the notions of dot product, linear transformations, determinants, eigenvalues, and quadratic forms. This is done in Chapters 2.0–2.7.

Then, the very same outline is used in Chapters 3.0–3.7 to present the linear algebra of three-dimensional space, so that the former ideas are reinforced while new concepts are being introduced.

In Chapters 4.0–4.2, we deal with geometry in $\mathbb{R}^n$ for $n \geq 4$. We introduce linear transformations and matrices in $\mathbb{R}^4$, and we point out that the step from $\mathbb{R}^4$ to $\mathbb{R}^n$ with $n > 4$ is now almost immediate. In Chapters 4.3 and 4.4, we treat systems of linear equations in n variables.

In the present edition, we have added Chapter 5 on vector spaces, Chapter 6 on inner products on a vector space, and Chapter 7 on symmetric $n \times n$ matrices and quadratic forms in n variables. Finally, in Chapter 8 we deal with three applications:

(1) differential systems, that is, systems of linear first-order differential equations;
(2) least-squares method in data analysis; and
(3) curvature of surfaces in $\mathbb{R}^3$, which are given as graphs of functions of two variables.

Except for Chapter 8, the student need only know basic high-school algebra and geometry and introductory trigonometry in order to read this book. In fact, we believe that high-school seniors who are interested in mathematics could read much of this book on their own. To read Chapter 8, students should have a knowledge of elementary calculus.

Acknowledgments

We would like to thank the many students in our classes, whose interest and suggestions have helped in the development of this book. We particularly thank Curtis Hendrickson and Davide Cervone, who produced the computer-generated illustrations in this new edition. Our special thanks go to Dale Cavanaugh, Natalie Johnson, and Carol Oliveira of the Brown University Mathematics Department for their assistance.

Contents

CHAPTER 1.0

Vectors in the Line

Analytic geometry begins with the line. Every point on the line has a real number as its coordinate and every real number is the coordinate of exactly one point. A *vector in the line* is a directed line segment from the origin to a point with coordinate x. We denote this vector by a single capital letter $\mathbf{X}$. The collection of all vectors in the line is denoted by $\mathbb{R}^1$.

We add two vectors by adding their coordinates, so if $\mathbf{U}$ has coordinate u, then $\mathbf{X} + \mathbf{U}$ has coordinate $x + u$. To multiply a vector $\mathbf{X}$ by a real number r, we multiply the coordinate by r, so the coordinate of $r\mathbf{X}$ is rx. The vector with coordinate zero is denoted by $\mathbf{0}$. (See Fig. 1.1.)

The familiar properties of real numbers then lead to corresponding properties for vectors in 1-space. For any vectors $\mathbf{X}, \mathbf{U}, \mathbf{W}$ and any real numbers r and s we have:

$\mathbf{X} + \mathbf{U} = \mathbf{U} + \mathbf{X}$.
$(\mathbf{X} + \mathbf{U}) + \mathbf{W} = \mathbf{X} + (\mathbf{U} + \mathbf{W})$.
For all $\mathbf{X}$, $\mathbf{0} + \mathbf{X} = \mathbf{X} = \mathbf{X} + \mathbf{0}$.
For any $\mathbf{X}$, there is a vector $-\mathbf{X}$ such that $\mathbf{X} + (-\mathbf{X}) = \mathbf{0}$.
$r(\mathbf{X} + \mathbf{U}) = r\mathbf{X} + r\mathbf{U}$
$(r + s)\mathbf{X} = r\mathbf{X} + s\mathbf{X}$
$r(s\mathbf{X}) = (rs)\mathbf{X}$
$1\mathbf{X} = \mathbf{X}$

We can define the length of a vector $\mathbf{X}$ with coordinate x as the absolute value of x, i.e., the distance from the point labelled x to the origin. We denote this length by $|\mathbf{X}|$ and we may write $|\mathbf{X}| = \sqrt{x^2}$. (We always understand this symbol to stand for the non-negative square root.) Then $\mathbf{0}$ is the

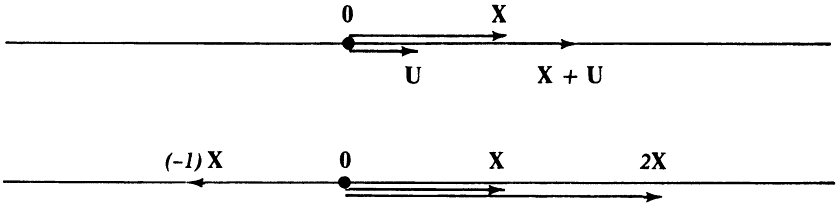

Figure 1.1

unique vector of the length 0 and there are just two vectors with length 1, with coordinates 1 and -1.

CHAPTER 2.0

The Geometry of Vectors in the Plane

Many of the familiar theorems of plane geometry appear in a new light when we rephrase them in the language of *vectors*. This is particularly true for theorems which are usually expressed in the language of analytic or coordinate geometry, because vector notation enables us to use a single symbol to refer to a pair of numbers which gives the coordinates of a point. Not only does this give us convenient notations for expressing important results, but it also allows us to concentrate on algebraic properties of vectors, and these enable us to apply the techniques used in plane geometry to study problems in space, in higher dimensions, and also in situations from calculus and differential equations which at first have little resemblance to plane geometry. Thus, we begin our study of linear algebra with the study of the geometry of vectors in the plane.

§1. The Algebra of Vectors

In vector geometry we define a *vector* in the plane as a pair of numbers $\begin{pmatrix} x \\ y \end{pmatrix}$ written in column form, with the *first coordinate* x written above the *second coordinate* y. We designate this vector by a single capital letter $\mathbf{X}$, i.e., we write $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. We can picture the vector $\mathbf{X}$ as an arrow, or directed line segment, starting at the origin in the coordinate plane and ending at the point with coordinates x and y. We illustrate the vectors $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, and $\mathbf{D} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ in Figure 2.1.

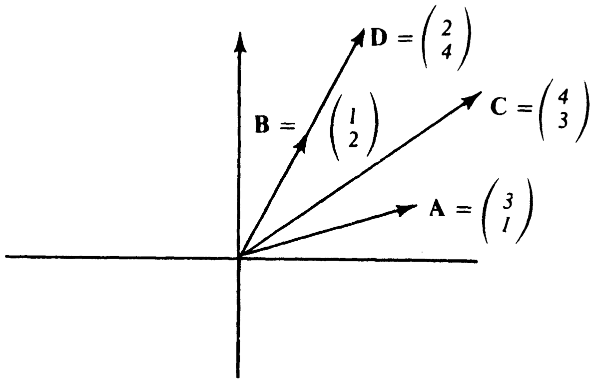

Figure 2.1

We *add* two vectors by adding their components, so if $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}$, we have

$$\mathbf{X} + \mathbf{U} = \begin{pmatrix} x + u \\ y + v \end{pmatrix}. \tag{1}$$

Thus, in the above diagram, we have $\mathbf{A} + \mathbf{B} = \mathbf{C}$, since

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + 1 \\ 1 + 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \mathbf{C}.$$

We *multiply* a vector $\mathbf{X}$ by a number r by multiplying each coordinate of $\mathbf{X}$ by r, i.e.,

$$r\mathbf{X} = r\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}. \tag{2}$$

In Fig. 2.1, $\mathbf{D} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\mathbf{B}$, and we also have $\mathbf{B} = \frac{1}{2}\mathbf{D}$.

Multiplying by a number r *scales* the vector $\mathbf{X}$ giving a longer vector $r\mathbf{X}$ if $r > 1$ and a shorter vector $r\mathbf{X}$ if $0 < r < 1$. Such multiplication of a vector by a number is called *scalar multiplication*, and the number r is called a *scalar*. If $r = 1$, then the result is the vector itself, so $1\mathbf{X} = \mathbf{X}$. If $r = 0$, then multiplication of any vector by $r = 0$ yields the *zero* vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, denoted by $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. If $\mathbf{X}$ is not the zero vector, then the scalar multiples of $\mathbf{X}$ all lie on a line through the origin and the point at the endpoint of the arrow representing $\mathbf{X}$. We call this line the *line along* $\mathbf{X}$. If $r > 0$, we get the points on the ray from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ through $\begin{pmatrix} x \\ y \end{pmatrix}$, while if $r < 0$, we get the points on the opposite ray. In particular, if $r = -1$, we get the vector $(-1)\mathbf{X} = (-1)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ which has the same length as $\mathbf{X}$ but the opposite

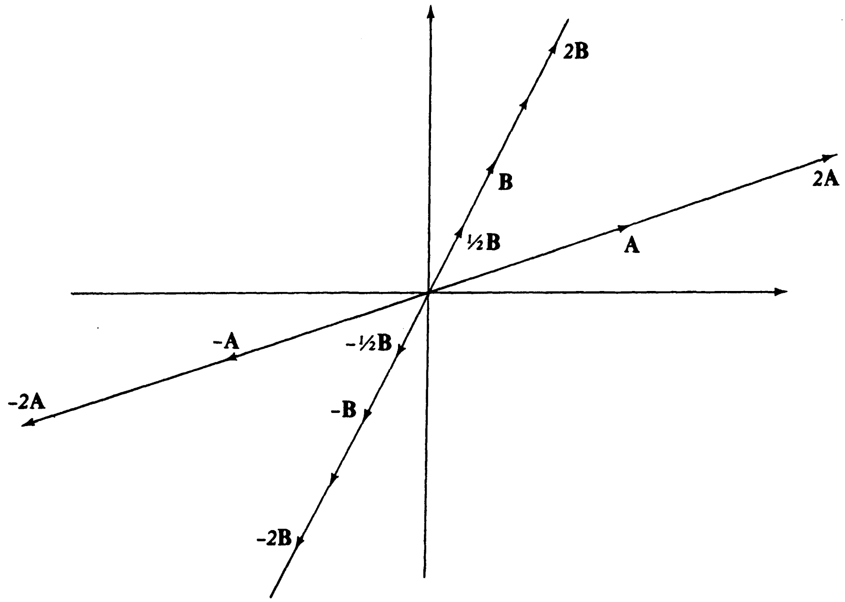

Figure 2.2

direction. We denote this vector by $-\mathbf{X} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ and we note that

$$\mathbf{X} + (-\mathbf{X}) = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} x + (-x) \\ y + (-y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}. \qquad (3\ )$$

We say that the vector $-\mathbf{X}$ is the *negative* of $\mathbf{X}$ or the *additive inverse* of $\mathbf{X}$.

In Figure 2.2, we indicate some scalar multiples of the vectors $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Two particularly important vectors are $\mathbf{E}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{E}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which we call the *basis vectors* of the plane. The collection of all scalar multiples $r\mathbf{E}_1 = r\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$ of $\mathbf{E}_1$ then gives the *first coordinate axis*, and the *second coordinate axis* is given similarly by $s\mathbf{E}_2 = s\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix}$. Since $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathbf{E}_1 + y\mathbf{E}_2$, we may express any vector $\mathbf{X}$ uniquely as a sum of one vector from the first coordinate axis and one vector from the second coordinate axis. Thus,

$$\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\mathbf{E}_1 + \mathbf{E}_2,$$

and, similarly, $\mathbf{D} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\mathbf{E}_i + 4\mathbf{E}_2$.

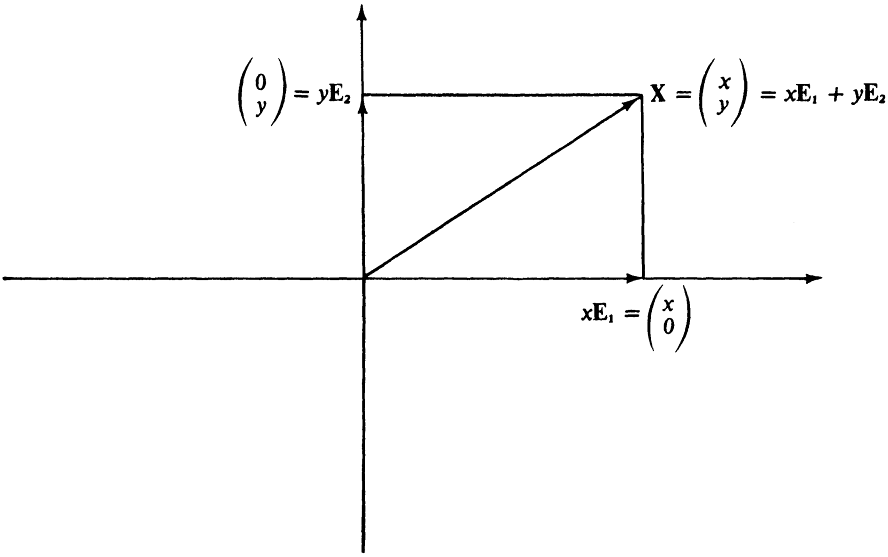

Figure 2.3

Writing a vector in this way expresses the point $\begin{pmatrix}x\\y\end{pmatrix}$ as the fourth vertex of a rectangle whose other three coordinates are $\begin{pmatrix}x\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\end{pmatrix}$, and $\begin{pmatrix}0\\y\end{pmatrix}$. (See Fig. 2.3.)

More generally, we may obtain a geometric interpretation of vector addition as follows. If we start with the triangle with vertices $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}x\\0\end{pmatrix}$, $\begin{pmatrix}x\\y\end{pmatrix}$

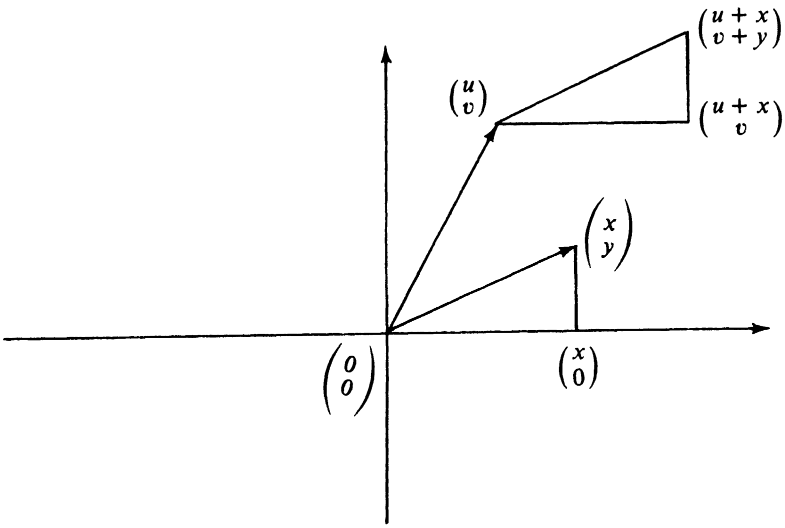

Figure 2.4

and move it parallel to itself so that its first vertex lies on $\begin{pmatrix} u \\ v \end{pmatrix}$, then the other two vertices lie on $\begin{pmatrix} u+x \\ v \end{pmatrix}$ and $\begin{pmatrix} u+x \\ v+y \end{pmatrix}$, respectively. (See Fig. 2.4.) Thus, the sum of the vectors $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}$ can be obtained by translating the directed segment from **0** to **X** parallel to itself until its beginning point lies at **U**. The new endpoint will represent **U** + **X**, and this will be the fourth coordinate of a parallelogram with **U**, **0**, and **X** as the other three vertices.

In our diagrams we have pictured addition of a vector **X** with positive coordinates, but a similar argument shows that the parallelogram interpretation is still valid if one or both coordinates are negative or zero.

By referring either to the coordinate description or the geometric description, we can establish the following algebraic properties of vector addition and scalar multiplication which are analogous to familiar facts about arithmetic of numbers:

(4)	$\mathbf{X} + \mathbf{U} = \mathbf{U} + \mathbf{X}$.	Commutative law for vectors
(5)	$(\mathbf{X} + \mathbf{U}) + \mathbf{A} = \mathbf{X} + (\mathbf{U} + \mathbf{A})$.	Associative law for vectors
(6)	There is a vector **0** such that $\mathbf{X} + \mathbf{0} = \mathbf{X} = \mathbf{0} + \mathbf{X}$ for all **X**.	Additive identity
(7)	For any **X** there is a vector $-\mathbf{X}$ such that $\mathbf{X} + (-\mathbf{X}) = \mathbf{0}$.	Additive inverse
(8)	$r(\mathbf{X} + \mathbf{U}) = r\mathbf{X} + r\mathbf{U}$.	Distributive law for vectors
(9)	$(r + s)(\mathbf{X}) = r\mathbf{X} + s\mathbf{X}$.	Distributive law for scalars
(10)	$r(s\mathbf{X}) = (rs)\mathbf{X}$.	Associative law for scalars
(11)	$1 \cdot \mathbf{X} = \mathbf{X}$ for each **X**.	

Note that it is possible for the parallelogram to collapse to a doubly covered line segment if we add two multiples of the same vector. In Fig. 2.5, we show the parallelograms for **B** + **B**, **A** + **B**, and **A** + (−**A**).

We can use the negative of a vector to help define the notion of *difference* **U** − **X** of the vectors **X** and **U**. (See Fig. 2.6.) We define

$$\mathbf{U} - \mathbf{X} = \mathbf{U} + (-\mathbf{X}),$$

so, in coordinates,

$$\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{U} - \mathbf{X} = \mathbf{U} + (-\mathbf{X}) = \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} u-x \\ v-y \end{pmatrix}.$$

Since $(\mathbf{U} - \mathbf{X}) + \mathbf{X} = \mathbf{U} + ((-\mathbf{X}) + \mathbf{X}) = \mathbf{U} + \mathbf{0} = \mathbf{U}$, we see that **U** − **X** is the vector we add to **X** to get **U**. Thus, if we move **U** − **X** parallel to itself until its beginning point lies on **X**, we get the directed line segment from **X** to **U**. Thus,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

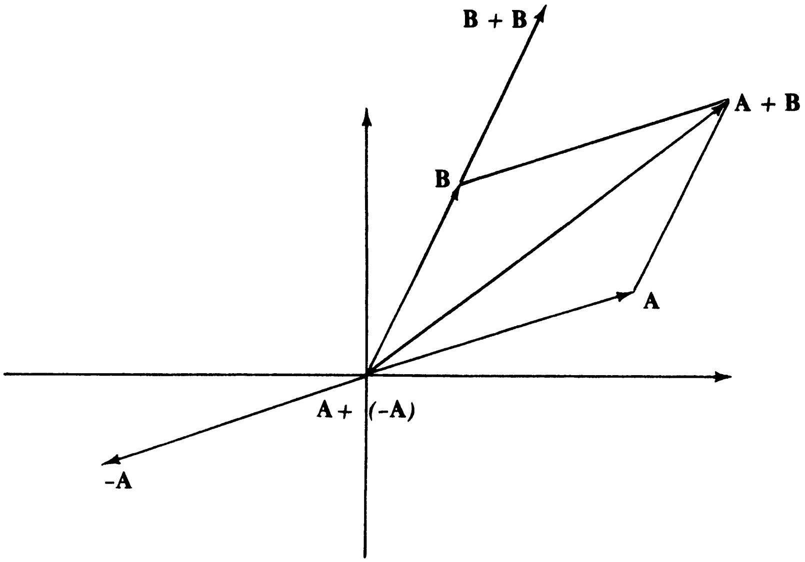

Figure 2.5

A pair of vectors **A**, **B** is said to be *linearly dependent* if one of them is a multiple of the other. If $\mathbf{A} = \mathbf{0}$, then the pair **A**, **B** is linearly dependent, since $\mathbf{0} = 0 \cdot \mathbf{B}$ no matter what **B** is. If $\mathbf{A} \neq \mathbf{0}$ and the pair **A**, **B** is linearly dependent, then $\mathbf{B} = t\mathbf{A}$ for some t. If $\mathbf{B} = \mathbf{0}$, then we use $t = 0$, but if **A** and **B** are both nonzero, we have $\mathbf{B} = t\mathbf{A}$ and $(1/t)\mathbf{B} = \mathbf{A}$, so each of the vectors is a multiple of the other.

If **A**, **B** is a linearly dependent pair of vectors and both **A** and **B** are nonzero, then the vectors $r\mathbf{A}$ for different values of r all lie on a line through the origin. The fact that **A**, **B** is a linearly dependent pair means that **B** lies on the line determined by **A**.

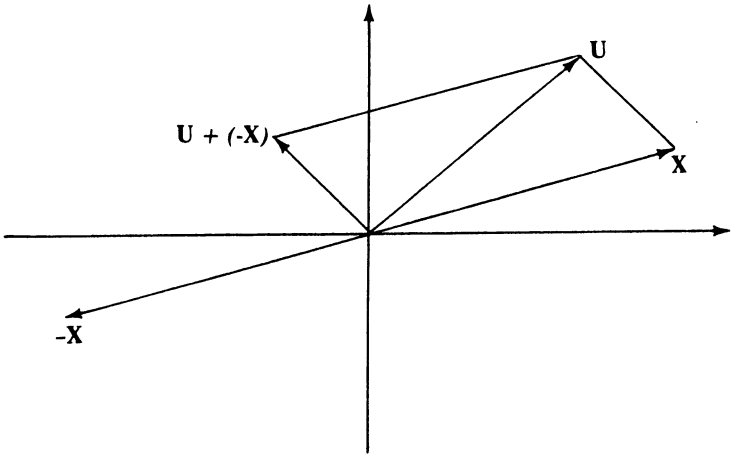

Figure 2.6

Exercise 1. For which choice of x will the following pairs be linearly dependent?

$$\text{(a) } \left(\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}\right), \quad \text{(b) } \left(\begin{pmatrix} x \\ x^2 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \end{pmatrix}\right),$$

$$\text{(c) } \left(\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ x \end{pmatrix}\right), \quad \text{(d) } \left(\begin{pmatrix} x \\ x^2 \end{pmatrix}, \begin{pmatrix} 1 \\ x \end{pmatrix}\right).$$

Exercise 2. True of false? If A is a scalar multiple of **B**, then **B** is a multiple of **A**.

Exercise 3. True or false? If **A** is a nonzero scalar multiple of **B**, then **B** is a nonzero scalar multiple of **A**.

Just as the multiples $t\mathbf{X}$ of a nonzero vector **X** give a description of a line through the origin, we may describe a line through a point **U** parallel to the vector **X** by taking the sum of **U** and all multiples of **X**. The line is then given by $\mathbf{U} + t\mathbf{X}$ for all real t. (See Fig. 2.7.)

For example, the line through $\mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ parallel to the vector $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is given by $\mathbf{X} = \mathbf{B} + t\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3t \\ t \end{pmatrix} = \begin{pmatrix} 2 + 3t \\ 1 + t \end{pmatrix}$. This is called the *parametric representation* of a line in the plane, since the coordinates $x = 2 + 3t$ and $y = 1 + t$ are given linear functions of the *parameter t*. Similarly, the line given by the parametric equation in coordinates $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 + 4t \\ 1 + 2t \end{pmatrix}$ can be written in vector form as $\mathbf{X} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4t \\ 2t \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \mathbf{A} + t\mathbf{D}$.

Exercise 4. Write an equation of the line through $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ parallel to the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

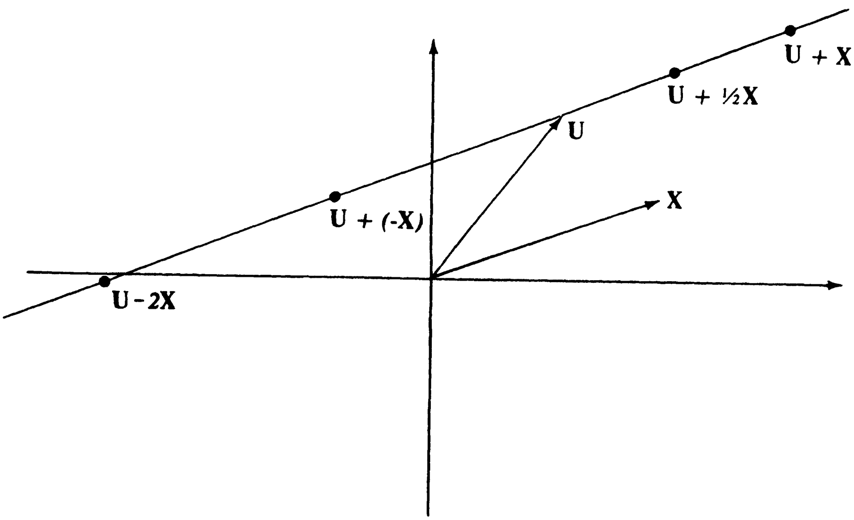

Figure 2.7

Exercise 5. Write an equation for the line through $\mathbf{A} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *Hint*: This line will go through $\mathbf{B}$ and be parallel to the vector $\mathbf{B} - \mathbf{A}$.

Exercise 6. Show that the parametric equation $\mathbf{X} = \mathbf{V} + t(\mathbf{U} - \mathbf{V})$ represents the line through $\mathbf{U}$ and $\mathbf{V}$ if $\mathbf{U}$ and $\mathbf{V}$ are any two vectors which are not equal.

By the Pythagorean Theorem, the distance from a point $\begin{pmatrix} x \\ y \end{pmatrix}$ to the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is $\sqrt{x^2 + y^2}$, and we define this number to be the *length* of the vector $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, written $|\mathbf{X}|$. For example, if $\mathbf{X} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then $|\mathbf{X}| = \sqrt{3^2 + 4^2} = 5$, while $|\mathbf{E}_1| = \left|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right| = 1$ and $|\mathbf{0}| = \sqrt{0^2 + 0^2} = 0$. Since the square root is always considered to be positive or zero, the length of a vector is never negative, and in fact $|\mathbf{X}|$ is positive unless $\mathbf{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

EXAMPLE 1. $|\mathbf{X} - \mathbf{U}| = \left|\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix}\right| = \left|\begin{pmatrix} x-u \\ y-v \end{pmatrix}\right| = \sqrt{(x-u)^2 + (y-v)^2}$.

For any scalar r, we have

$$|r\mathbf{X}| = \left|\begin{pmatrix} rx \\ ry \end{pmatrix}\right| = \sqrt{(rx)^2 + (ry)^2} = \sqrt{r^2x^2 + r^2y^2} = |r|\sqrt{x^2 + y^2} = |r||\mathbf{X}|.$$

Thus, the length of a scalar multiple of a vector is the length of the vector multiplied by the absolute value of the scalar. For example, $|-5\mathbf{X}| = |-5|\,|\mathbf{X}| = 5|\mathbf{X}|$.

Exercise 7. Show that the midpoint of the segment joining points $\mathbf{X}$ and $\mathbf{U}$ is $\frac{1}{2}(\mathbf{X} + \mathbf{U})$, i.e. show that this point lies on the line through $\mathbf{X}$ and $\mathbf{U}$ and is equidistant from $\mathbf{X}$ and $\mathbf{U}$.

If $\mathbf{X} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $\mathbf{X} \neq \mathbf{0}$, so we may scale by the reciprocal $(1/|\mathbf{X}|)$ to get a vector $(1/|\mathbf{X}|)\mathbf{X}$. This vector lies along the ray from $\mathbf{0}$ to $\mathbf{X}$ and it has length equal to 1 since

$$\left|\left(\frac{1}{|\mathbf{X}|}\right)\mathbf{X}\right| = \left|\frac{1}{|\mathbf{X}|}\right||\mathbf{X}| = \frac{1}{|\mathbf{X}|}|\mathbf{X}| = 1.$$

The vectors of length 1 are called *unit vectors*, and they are represented by the points on the unit circle in the coordinate plane. The vector $(1/|\mathbf{X}|)\mathbf{X}$ is represented by the point where the ray from $\mathbf{0}$ to $\mathbf{X}$ intersects this unit circle. (See Fig. 2.8).

Any vector on the unit circle may be described by its angle θ from the ray along the positive x-axis to the ray along the unit vector. We call θ the *polar angle* of the vector. We may then write the unit vector using trigonometric functions as $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.