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357

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Degeneracy Graphs and Simplex Cycling

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Foreword

The first monograph on "Degeneracy Problems" appeared 1986 as "Degeneracy Graphs and the Neighbourhood Problem" by H.-J. Kruse as No 260 of Lecture Notes. The contents of that book was based upon a simple problem of degeneracy in linearly constraint optimization problems which has been posed in 1976. Since then the literature concerning degeneracy grew and a part of the corresponding research throughout the world has been devoted to the theory and application of degeneracy graphs. It turned out that the better should be the use of degeneracy graphs to solve practical problems of various kind the more theoretical knowledge about such graphs is needed. Just to mention one of such problems, let us remind of the cycling of the simplex method. This phenomenon is known since the beginning of the 50-ies and caused some trouble. Several "anticycling" methods have been elaborated as the time passed and even in the late 80-ies new proposals have been published. Strange enough, there has been no publication in which a trial was made to explain this phenomenon, i.e. to pose the question "When or under which conditions simplex-cycling occurs?".

The author of this book makes a valuable contribution to the theory of the degeneracy graphs and answers the above question.
Based on his results the research into degeneracy and degeneracy graphs can now turn to new unsolved degeneracy problems, which would remain unsolvable without the knowledge presented in this book.

Hagen, May 1991 Tomas Gal

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Bochum, May 1991 Peter Zörnig

To my parents
who could not live to see
the publication of this book

Contents

Index of symbols

$A, \bar{A}$	(enlarged) matrix of coefficients
b	right-hand side vector
$B, B_{j_1,\ldots,j_\sigma}$	basis (with index set $\{j_1, \ldots, j_\sigma\}$)
B^0	basis set of the vertex x^0
$B \leftrightarrow B^*$	possibility to pass
$B \overset{+}{\leftrightarrow} B^*$	from B to B^* in one pivot step
$B \overset{-}{\leftrightarrow} B^*$	with an (arbitrary, positive, negative) pivot element
β_i	cf. Section 4.3.1
$c, \bar{c}$	(enlarged) vector of objective function coefficients
c_B	vector of objective function coefficients assigned to basis B
C	cycle of a degeneracy graph or simplex cycle
$d, \bar{d}$	(enlarged) vector of objective function coefficients belonging to the reduced linear optimization problem
$d(v, w)$	distance between the nodes v and w of a graph
$d(G)$	diameter of the graph G
$\tilde{d}$	cf. Section 4.3.1
$D_{j_1,\ldots,j_\sigma}$	subdeterminants of the initial tableau belong-
$\bar{D}_{j_1,\ldots,j_\sigma,\nu}$	ing to a reduced linear optimization problem
$\bar{D}^i_{j,k}, D^i_j$	(cf. Sections 4.4.1 and 5.2.5.1)
$\mathcal{D}(\mathcal{S})$	σ-normal representation system of $\mathcal{S}$
Δz_j	relative cost coefficients
δ	minimum degree of a graph
e_u	u-th unit vector
E	edge set of a graph
$E_\nu, E_{a,b}$	(constraint-)hyperplane
$g(v)$	degree of the node v
$G_1 \cong G_2$	G_1 is isomorphic to G_2
$G(X)$	representation graph of X
$G'(X)$	graph of the polytope X
G^0, G^0_+, G^0_-	general, positive and negative degeneracy graph of x^0
G_Y, G^+_Y, G^-_Y	general, positive and negative canonical degeneracy graph of x^0

$G_{Y,d}, G^+_{Y,d}, G^-_{Y,d}$	general, positive and negative LP-degeneracy graph
$G^{\sigma,n}$	complete $\sigma \times n$-index graph
$H_\nu, H_{a,b}$	halfspace of $\mathbb{R}^n$
int X	interior of X
I	index set
I_m	$m \times m$-unit matrix
j^+, j^+_ν	index of the entering basic variable
K	edge of a graph
K_C	cone induced by C
$(K_1, \ldots, K_n)$	edge path from K_1 to K_n
$K(p_1, \ldots, p_r)$	complete r-partite graph
$L(G)$	line graph of G
$\mathbb{N}$	set of natural numbers
ω, ω'	node-/edge-connectivity
$p(n)$	number of (unordered) partitions of n
P_C	point set induced by C
$\mathbb{R}, \mathbb{R}^n$	set of real numbers/ vectors with n real components
$\mathbb{R}^{m \times n}$	set of $m \times n$-matrices with real elements
$\mathcal{S} = \{s_1, \ldots, s_p\}$	σ-homogeneous set system
$\bar{\mathcal{S}}$	complementary system of $\mathcal{S}$
$< \mathcal{S} >$	subgraph of $G^{\sigma,n}$ induced by $\mathcal{S}$
S_C	star-shaped graph
σ	degeneracy degree
$T, T_{j_1,\ldots,j_\sigma}$	tableau (corresponding to basis $B_{j_1,\ldots,j_\sigma}$)
$\mathcal{T} = \{t_1, \ldots, t_q\}$	(σ-normal) representation system
$\mathcal{T}_Y$	set system induced by Y
u	vector of slack variables
U	number of nodes of a graph
U'	number of edges of a graph
$U(x^1, \ldots, x^k)$	vector space generated by $x^1, \ldots, x^k$
$U(x^0; x^1, \ldots, x^k)$	affine subspace parallel to $U(x^1, \ldots, x^k)$
v	node of a graph
$(v_1, \ldots, v_n)$	trail from v_1 to v_n
V	node set of a graph
V^*	node set of $L(G)$
$V^*_{i,j}$	component of a partition of V^*

$V(x,y)$	closed segment between x and y
W	path of a graph
$x, \bar{x}$	(enlarged) vector of variables
$x_B, x_B^{(0)}$	(complete) basic solution of the basis B
x^0	vertex of X
$X, \bar{X}$	solution set of a linear optimization problem
y^j	column of $\bar{Y}$
$\tilde{y}^j$	cf. Section 4.3.1
$Y, \bar{Y}$	(enlarged) matrix of coefficients of a reduced linear optimization problem
z	objective function value

1. Introduction

Problems with linear constraints are of special importance in the topic of mathematical optimization. The feasible solution set X represents a convex polyhedral set. In practice the case that X contains degenerate (overdetermined) vertices, occurs very often[1]. Degeneracy involves many different problems, not only in connection with the determination of an optimal solution by a "vertex searching" procedure but also in postoptimal analysis (cf. Chapter 2). Shortly after Dantzig (1951) published the simplex method, a number of papers on the degeneracy phenomenon appeared[2]. Recent articles[3] demonstrate that many questions are still open[4]. The reason for all kinds of degeneracy problems is the fact that a great number of bases is associated with a degenerate vertex[5]. The extensive literature shows that degeneracy problems have always been treated separately from each other. The idea of studying all these problems from a common point of view led to the investigation of so-called degeneracy graphs (cf. Section 3.1). These graphs represent appropriate tools for describing the complex structure of the basis or tableau set associated with a degenerate vertex and thus for revealing the theoretical background of the emergence of different degeneracy phenomena. Initial investigations of these graphs were carried out by Gal (1978, 1985) in connection with the problem of determining all neighbours of a degenerate vertex. Degeneracy graphs also render valuable help in the solution of different other kinds of degeneracy problems. Based on the theory of degeneracy graphs it has already been possible to develop procedures to solve the so-called neighbourhood

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[1] Cf. e.g. Greenberg (1986:636, 650), Perold(1980:240) and Section 3.1.1.

[2] Cf. e.g. Charnes (1952), Hoffmann (1953), Nelson (1957).

[3] Cf. Balinski et al. (1986), Cameron (1987), Cirina' (1985,1989), Clausen (1987), Fourer (1988), Greenberg (1986), Hattersley/Wilson (1988), Horst et al. (1988), Megiddo (1986), Mlynarovic (1988), Nygreen (1987), Ramesh et al. (1987), Ryan/Osborne (1988), Sherali/Dickey (1986), Vörös (1987).

[4] Perhaps the "competing procedures" of Khachian (1979) and Karmarkar (1984) have motivated scientists to reflect on the "classical" simplex method and different aspects of degeneracy connected with it.

[5] Cf. Dantzig (1966:Ch. 10), Gal (1985), Kruse (1986: Ch.4) among others.

problem and to achieve better understanding of the theoretical foundation of sensitivity analysis and the determination of shadow prices under degeneracy. For the solution of the neighbourhood problem the fact that every positive degeneracy graph contains an N-tree[6] is fundamental; for sensitivity analysis and the determination of shadow prices the question whether optimum graphs[7] are connected is decisive. Surveys of theory and application of degeneracy graphs are given in Gal (1988) and Gal/Kruse/Zörnig (1986,1988).

The present publication starts from the results above. It has essentially two aims:

1) In order to make the use of degeneracy graphs more effective, it is desirable to know as much as possible about their structural properties. Therefore the theory of degeneracy graphs will be developed generally.
2) Degeneracy graphs will be used to explain a special degeneracy problem, namely the simplex cycling in linear optimization. The intention here is not to develop a further anticycling rule, but to investigate the following unsolved question: "Under which conditions does the phenomenon of simplex cycling occur at all?" The answer to this question is of theoretical as well as practical importance.

The present book is divided into five chapters. Below a summary of different degeneracy problems in the topic of mathematical optimization is to be found. In Chapter 3 the theory of degeneracy graphs is developed. A characterization of these graphs is derived which especially implies certain structural properties (numbers of nodes and edges, connectivity etc.). Applications of these results are also mentioned (Section 3.2.3). The last two chapters deal with the problem of simplex cycling. Chapter 4 presents diverse concepts to explain simplex cycling using degeneracy graphs. For each case necessary and sufficient conditions for the occurrence of simplex cycling are derived. Among other things it will become evident that simplex cycling occurs

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[6] Cf. e.g. Kruse (1984,1986) and Gal/Kruse (1984).

[7] Cf. Kruse (1987), Piehler (1988), Piehler/Kruse (1989) and Section 2.3.

if and only if the positive degeneracy graph can be enlarged to a "star-shaped" graph contained in the LP-degeneracy graph. Based on these results Chapter 5 evolves procedures to construct cycling examples. They permit carrying out certain tests to clarify different practically oriented questions concerning simplex cycling. For example the efficiency of anticycling rules can be tested by applying them to a set of constructed cycling examples[8].

2. DEGENERACY PROBLEMS IN MATHEMATICAL OPTIMIZATION

Simplex cycling in linear optimization is the most popular degeneracy problem[9] (cf. Ch. 4). Moreover, the degeneracy phenomenon causes problems in many other fields of mathematical optimization with regard to convergence and the efficiency of the algorithms determining the optimal solution. The following selection[10] of problems will provide an impression of the extension of the topic "degeneracy" and its practical importance.

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[8] Further applications of constructed cycling examples are mentioned in Chapter 5.1.

[9] This problem was already investigated in the fifties (cf. Beale (1955), Charnes (1952), Dantzig/Orden/Wolfe (1955), Hoffmann (1953)).

[10] Numerous references are adduced from Mathies (1989). Further fields in which degeneracy causes difficulties are: Neighbourhood problem (Gal (1985) and Kruse (1984,1986)), methods for solving systems of linear inequalities or for determining the vertices of a convex polyhedral set (Dyer/Proll (1980,1982), Sherali/Dickey (1986), Wallace (1985)), parametric programming (Gal (1979), Ritter (1984)), vector maximization (Philip (1977), Proll (1987)), portfolio analysis (Vörös (1987)), infinite-dimensional linear optimization (Nash (1985)), piecewise-linear optimization (Fourer (1988), Ruszczynski (1986)), global optimization (Horst et al.(1988)), fixed cost transportation (Ahrens/Finke (1975), Mc Keown (1978)), linear minimax optimization (Ahuja (1985)), linearly constrained optimization (Calmai/More'(1987, Section 4)), linear optimization with variable upper bounds ((Todd (1982)), reduced linear optimization (Tomlin/ Welch (1983)).

2.1 CONVERGENCE PROBLEMS IN THE CASE OF DEGENERACY

2.1.1 CYCLING IN LINEAR COMPLEMENTARITY PROBLEMS

Linear complementarity problems occur in

a) the solution of quadratic optimization problems,

b) bimatrix games[11].

To a: In order to solve a *quadratic optimization problem* of the form[12]

$$\left.\begin{array}{rl} \min z = & \frac{1}{2}x^T Qx + q^T x \\ \text{s.t.} & \\ & Ax \leq b \\ & x \geq 0, \end{array}\right\} \qquad (2.1.1.1)$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric, positive definite matrix and $A \in \mathbb{R}^{m \times n}, q, x \in \mathbb{R}^n, b \in \mathbb{R}^m$, the Kuhn-Tucker conditions are generally used. They are necessary and sufficient for an optimal solution of (2.1.1.1) and can be represented in the form

$$\left.\begin{array}{ll} w - Mz & = r \\ w, z & \geq 0 \\ w^T z & = 0 \end{array}\right\} \qquad (2.1.1.2)$$

where $M \in \mathbb{R}^{p \times p}, w, z, r \in \mathbb{R}^p$. The variables w, z must be determined such that (2.1.1.2) is fulfilled (M, r fixed). The system (2.1.1.2) is called a *linear complementarity problem.* The solution of (2.1.1.1) can be obtained immediately from the solution of (2.1.1.2).

To b: A *Bimatrix game* (i.e. a nonzero-sum two person game) can also be formulated as a linear complementarity problem (2.1.1.2). The solution represents the equilibrium point[13].

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[11] Cf. Lemke (1965:681).

[12] Cf. Horst (1987:270).

[13] Cf. Bitz (1988:23ff.), Schwödiauer (1987:11ff.).

In order to solve (2.1.1.2) the algorithm of Lemke (1965) is generally used (cf. also Horst (1987:377ff.)). This procedure starts with a tableau of the form

$$w|z_0| - M|r \qquad (2.1.1.3)$$

and solves (2.1.1.2) by appropriate pivot steps. Kostreva (1979) detected that Lemke's algorithm cycles for certain constructed degenerate problems. Cycling is also possible when Keller's (1973) algorithm is applied (cf. Chang/Cottle (1980)) which is a pivoting method starting with a "symmetric" modification of (2.1.1.3) to solve (2.1.1.2). However, the use of Bland's (1977) anticycling rule guarantees the convergence of Keller's algorithm.[14] .

2.1.2 CYCLING IN NETWORK PROBLEMS

A *network problem* or *transshipment problem* is a linear optimization problem of the form (cf. Bixby/Cunningham (1980) a.o.):

$$\left.\begin{array}{lll} \min z = & c^T x & \\ \text{s.t.} & & \\ & Nx & = b \\ & x & \geq 0, \end{array}\right\} \qquad (2.1.2.1)$$

where $c, x \in \mathbb{R}^n, b \in \mathbb{R}^m$ are vectors and $N \in \mathbb{R}^{m \times n}$ is a so-called node-arc incidence matrix, the elements being +1,-1 or 0 (cf.also Solow (1984:348)). *Transportation* and *assignment problems* are special cases of (2.1.2.1) which play an important role in practice. The former are generally solved by procedures based on the vertex searching method (MODI method, Stepping-Stone algorithm) developed by Dantzig (1951a) and Charnes/Cooper (1954)[15]. Since transportation problems

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[14] Cf. Chang/Cottle (1980:128ff.).

[15] The only difference between these procedures consists in the computation of opportunity costs (cf.Domschke (1981:102), Ohse (1987:267)).

in practice are often "highly degenerate"[16], cycling can occur using the above methods. Gassner (1964) pointed out that the MODI method cycles, when it is applied to specially constructed $n \times n$-assignment problems[17] with $n \geq 4$. Subsequently Madan Lal Mittal (1967:176ff.) developed a modification of the MODI method that insures the convergence. A finite variant of the simplex method for solving network problems which uses "strongly feasible trees" was elaborated by Cunningham (1976). Especially in solving transportation problems, this procedure coincides with the method of Barr/Glover/Klingman (1977,1978). A general cycling example for the network simplex method was introduced by Cunningham (1979)[18].

2.1.3 CYCLING IN BOTTLENECK LINEAR PROGRAMMING

A *bottleneck linear programming problem* is of the form

$$\left.\begin{array}{rl} \min z = & \max\{c_j \mid j \in \{1,\ldots,n\}, x_j > 0\} \\ \text{s.t.} & \\ & Ax = b \\ & x \geq 0, \end{array}\right\} \quad (2.1.3.1)$$

with $c = (c_1,\ldots,c_n)^T, x = (x_1,\ldots,x_n)^T \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ (cf. Frieze (1975:871)). In analogy to network problems, model (2.1.3.1) includes the *bottleneck transportation problem* and the *bottleneck assignment problem* as special cases. Different practical problems can be represented by (2.1.3.1), e.g. the problem of parallel transportation of perishable goods, which have to reach their destinations as fast as possible (cf. Garfinkel/Rao (1976:291)).

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16 Cf. Boenchendorf (1987:87), Madan Lal Mittal (1967:175).

17 Note that the assignment problem is the most degenerate form of the transportation problem (cf. Akgül (1987)).

18 Cunningham/Klincewicz (1983) showed that any cycling example for the network simplex method has a length of at least ten.

In order to solve problem (2.1.3.1), simplex type algorithms have been developed[19]. Though no cycling example for bottleneck linear programming is mentioned explicitly in literature, the possibility of cycling can not be excluded. Seshan/Achary (1982:349) propose the application of the "lexicographic rule" of Dantzig/Orden/Wolfe (1955) to insure the convergence.

2.1.4 CYCLING IN INTEGER PROGRAMMING

A pure *integer programming problem* is of the form

$$\left.\begin{array}{lll} \max z = & c^T x & \\ \text{s.t.} & & \\ & Ax & \leq b \\ & x & \geq 0 \\ & x & \text{integer vector} \end{array}\right\} \qquad (2.1.4.1)$$

where $x, c \in I\!R^n, A \in I\!R^{m \times n}, b \in I\!R^m$, and all elements of c, A, b are also integers. The so-called cutting plane algorithms (cf. e.g. Gomory (1963), Young (1968)) represent an important class of solution methods for (2.1.4.1). The common principle of these procedures is the following[20]: First of all problem (2.1.4.1) is solved without the integer requirements. If the solution is not an integer vector, another cutting plane is added to system (2.1.4.1) which "cuts off" the just found solution without changing the set of feasible solutions. Now the new problem has to be solved without integer requirements, etc. In (primal) cutting plane algorithms the case of degeneracy (i.e. the right-hand side of the "cutting constraint" equals 0; cf. Burkard (1972:237ff.)) involves specific difficulties which anticycling rules can not eliminate[21]. In general linear optimization an anticycling rule insures the termination of the procedure, since

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[19] Different solution methods are surveyed in Seshan/Achary (1982:347).

[20] Cf. Burkard (1987:379ff.).

[21] Degeneracy in integer programming not only involves the danger of cycling but also reduces the efficiency of solution methods (cf. Balas (1971), Glover (1968:729), Nygreen (1987), Tomlin (1970)).

a) only a restricted number of feasible basic solutions associated with vertices exists

and

b) the anticycling rule avoids the repetition of basic solutions.

In cutting plane algorithms it is generally impossible to determine upper bounds for the number of cutting planes or the number of vertices of the solution set. Hence assumption a) is not fulfilled, i.e. the termination is not guaranteed even if the repetition of basic solutions is impossible. Young (1968) developed appropriate modifications to insure the termination of the primal cutting plane algorithm in the case of degeneracy.

2.2 EFFICIENCY PROBLEMS IN THE CASE OF DEGENERACY

2.2.1 EFFICIENCY LOSS BY WEAK REDUNDANCY

A constraint of a linear optimization problem is called redundant if it can be omitted without effecting the set of feasible solutions (cf. Karwan et al.(1983:1)). If a redundant constraint is active for any vertex of the solution set, it is called weakly redundant[22]. In this case the corresponding vertex is degenerate. Thus weak redundancy is a special case of degeneracy occuring especially if the system of constraints consists of different subsystems which have to be preserved individually[23] or if the solution method requires the addition of further constraints (e.g. cutting plane methods in integer programming)[24]. The loss in efficiency caused by (weak) redundancy may be considerable. Since redundant constraints have to be transformed in each iteration, they

[22] Cf. Gal (1983,1987:160), Kruse (1984:12), Nelson (1957:405f) and Section 3.1.1.

[23] Cf. Tomlin/Welch (1983:233f).

[24] Cf. Müller-Merbach (1973:386f).