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Preface

This is the fourth volume in the series "Mathematics in Industrial Problems." The motivation for these volumes is to foster interaction between Industry and Mathematics at the "grass roots"; that is, at the level of specific problems. These problems come from Industry: they arise from models developed by the industrial scientists in ventures directed at the manufacture of new or improved products. At the same time, these problems have the potential for mathematical challenge and novelty.

To identify such problems, I have visited industries and had discussions with their scientists. Some of the scientists have subsequently presented their problems in the IMA Seminar on Industrial Problems. The book is based on questions raised in the seminar and subsequent discussions. Each chapter is devoted to one of the talks and is self-contained. The chapters usually provide references to the mathematical literature and a list of open problems which are of interest to the industrial scientists. For some problems partial solution is indicated briefly. The last chapter of the book contains a short description of solutions to some of the problems raised in the third volume, as well as references to papers in which such solutions have been published.

The speakers in the seminar on Industrial Problems have given us at the IMA hours of delight and discovery. My thanks to John Ockendon (Oxford University), Sanjay S. Patel (AT&T Bell Laboratories), Richard Olmsted (3M), Andrew Ogielski (Bellcore), Peter Blakey (Motorola, Inc.), Charles Rennolet (FMC), Edward Bissett (General Motors), Leonard Borucki (Motorola), Yitzhak Shnidman (Eastman Kodak), David Ross (Eastman Kodak), Pat Hagan (Los Alamos National Laboratory), Blaise Morton (Honeywell), Michael Honig (Bell Communications Research), Claude Greengard (IBM), Kam–Chuen Ng (Eastman Kodak), Carl Nelson (Alliant Techsystems Inc.), Gary Strumolo (Ford), Alan E. Ames (Polaroid), Erich Wimmer (Cray Research) and Robert Goor (General Motors).

Patricia V. Brick typed the manuscript and Stephen Mooney drew the figures; they did a superb job. Thanks are also due to the IMA staff Kathy Boyer, Ceil McAree, Mary Saunders, Kaye Smith, Kathi Polley Stephan Skogerboe, Joan Felton, Lisa Somers, Skye Johnson, Bob Gates, Jim McDonald, Jon Buerge, Paul Ewing and Marise Widmer, for creat-

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Avner Friedman
Director
Institute for Mathematics
and its Applications
June 14, 1991

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Real-world free boundary problems

The Mathematical Study Group is a group of Applied Mathematicians at Oxford University, England, who have been interacting with British industry over the past 22 years. Their interaction includes modeling of industrial processes as well as developing mathematical analysis of the models. One of the leading mathematicans of the Oxford Study Group is John Ockendon. On September 14, 1990 he gave a talk describing several problems that have emerged in recent years in hypersonic flow, porous medium flow, ship hydrodynamics, superconductivity, and contact problems. The common feature to these problems is that they are free boundary problems. Ockendon's presentation included a list of challenging open problems. The following write-up is based on his talk, preprints and notes.

1.1 Hypersonic flow

Hypersonic flow is one for which some typical Mach number M is large ($M \gg 1$). For a good introduction the reader is referred to the books [1] [2]. There is a revived interest in the modeling of hypersonic gas dynamics. This requires reliable mathematical results to test computational algorithms. For the sake of simplicity we shall ignore here the important effects of viscosity and chemical reactions (see, however, [3] for a study which includes the latter effects). When M becomes large, the key phenomenon is the shrinkage of gas downstream of a shock wave into a "shock layer" of thickness $O\left(M^{-2}, \dfrac{\gamma-1}{\gamma+1}\right)$ where γ is the specific heat ratio which decreases from 1.4 as M increases; it will be assumed later on that

$$\varepsilon \equiv \frac{\gamma-1}{\gamma+1} \ll 1 \; ; \tag{1.1}$$

this assumption though is physically somewhat questionable.

We consider here the 1-dimensional problem of a piston motion, with $x = x_p$ and $x = x_s$ the positions of the piston and shock respectively; see Figure 1.1

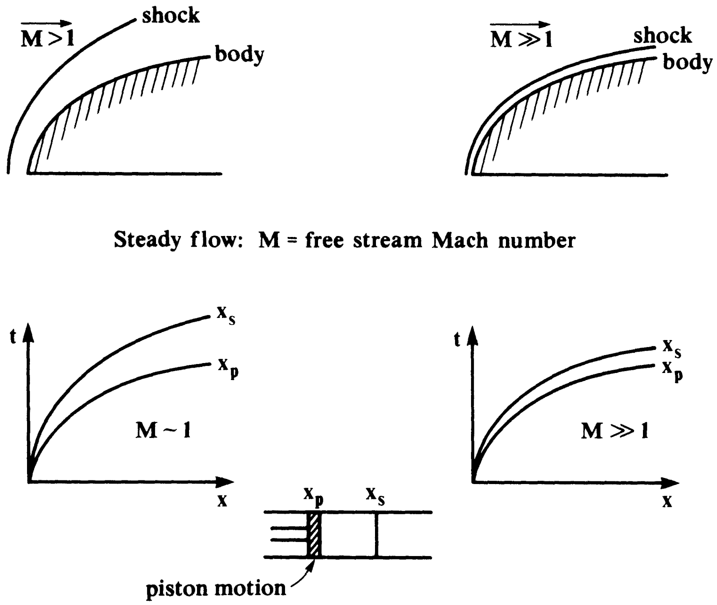

FIGURE 1.1

The Eulerian equations for the density ρ, pressure p and velocity u are:

$$\frac{d\rho}{dt} + \rho\frac{\partial u}{\partial x} = 0 \qquad \text{(conservation of mass)}, \tag{1.2}$$

$$\rho\frac{du}{dt} + \frac{\partial p}{\partial x} = 0 \qquad \text{(conservation of momentum)}, \tag{1.3}$$

$$\frac{de}{dt} + p\frac{d}{dt}\frac{1}{\rho} = 0 \tag{1.4}$$

where $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x}$, $e = \dfrac{p}{(\gamma - 1)\rho}$ is the internal energy, and the specific heat ratio γ is assumed constant. Then

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \; .$$

We shall shortly make the assumptions that $M \to \infty$ and γ is near 1, so that (1.1) holds.

Using the fact that the gas is undisturbed in $x > x_s(x)$ with pressure p_0 and density ρ_0, the Rankine–Hugoniot conditions along the shock $x = x_s(t)$

are

$$\frac{\rho}{\rho_0} = \frac{\gamma+1}{\gamma-1+2/M^2}\ , \tag{1.5}$$

$$\frac{p}{p_0} = \frac{2\gamma M^2-(\gamma-1)}{\gamma+1}\ , \tag{1.6}$$

$$u = \frac{2\dot{x}_s}{\gamma+1}\left(1-\frac{1}{M^2}\right) \tag{1.7}$$

where p_0 is constant and $M^2 = \rho_0\ \dot{x}_s^2/\gamma p_0$; ρ_0 need not be constant.

We introduce the particle path function ξ, a function satisfying $\partial\xi/\partial x = \rho, \partial\xi/\partial t = -\rho u$, and consider ξ as an independent variable. Then

$$x = x_p(t) \qquad \text{at} \quad \xi = 0$$

where $x_p(t)$ is the position of the piston at time t. With x, p, ρ as dependent functions of ξ, t, the equations of gas dynamics become

$$\frac{\partial x}{\partial \xi} = \frac{1}{\rho}\ , \tag{1.8}$$

$$\frac{\partial^2 x}{\partial t^2} + \frac{\partial p}{\partial \xi} = 0\ , \tag{1.9}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{1.10}$$

and, in addition to (1.5), (1.6),

$$x = x_s(t) \quad \text{at} \quad \xi = \int_0^{x_s} \rho_0 dx\ .$$

Letting $M \to \infty$ in (1.5), (1.6) we get

$$\rho = \frac{\gamma+1}{\gamma-1}\ \rho_0, \quad p = 2\rho_0\dot{x}_s^2/(\gamma+1),\ x = x_s(t) \quad \text{at} \quad \xi = \int_0^{x_s} \rho_0 dx\ . \tag{1.11}$$

We now use (1.1): we take $\rho_0 = 1$ for simplicity, and develop (1.8), (1.9), (1.10) and (1.11) into powers of ε, and compute formally the lowest order terms. Writing $\rho = \varepsilon^{-1}\ \overline{\rho}$, we obtain, to lowest order in ε (for details, see [3]):

$$\frac{\partial^2 x}{\partial t^2} + \frac{\partial p}{\partial \xi} = 0\ , \quad \frac{p}{\overline{\rho}} = G(\xi), \quad \frac{\partial x}{\partial \xi} = \frac{\varepsilon}{\overline{\rho}}$$

where $G(x_s) = \dot{x}_s^2$. Eliminating p we obtain for the unknown function $x = x(\xi, t)$ the nonlinear hyperbolic equation

$$\left(\frac{\partial x}{\partial \xi}\right)^2 \frac{\partial^2 x}{\partial t^2} = \varepsilon \left(G(\xi)\frac{\partial^2 x}{\partial \xi^2} - G'(\xi)\frac{\partial x}{\partial \xi}\right) \text{ if } 0 < \xi < x_s(t),\ t > 0 \quad (1.12)$$

together with the shock conditions

$$x = x_s(t)\ ,\ \frac{\partial x}{\partial \xi} = \varepsilon \quad \text{on} \quad \xi = x_s(t), \quad t > 0 \quad (1.13)$$

and the boundary condition

$$x = x_p(t) \qquad \text{at} \qquad \xi = 0\ . \quad (1.14)$$

The function $G(\xi)$ depends on the free boundary $\xi = x_s(t)$ according to

$$G(x_s(t)) = \dot{x}_s(t)^2\ . \quad (1.15)$$

We expect $x_s(t)$ to increase monotonically in t and thus $G(\xi)$ will be well defined.

If $x_p(t) = t^\alpha$ then a similarity solution exists with $x_s(t) = At^\alpha$ provided $\alpha > \frac{2}{3}$ (see [3] and the references given there); the solution breaks down at $\alpha = \frac{2}{3}$.

Conjecture: If $x_p(t) = t^\alpha$ and $\alpha \leq 2/3$ then there does not exist a solution of (1.12)–(1.15).

The case $\alpha = 2/3$ corresponds to the so called "blast wave" [4] [5] which models an instantaneous finite energy release.

Problems. (1) Prove the above conjecture.

(2) If $x_p(t) = t^\alpha f(t)$ where $\alpha > \frac{2}{3}$ and $f(t)$ is a smooth positive function for $t \geq 0$ then there exists a solution to the hyperbolic free boundary problems (1.12)–(1.15).

Other hyperbolic problems for chemically reacting flows are described in [3], and also lead to free boundary problems.

1.2 Problems with free boundaries close to fixed boundaries

Free boundary problems in which the free boundary is near a known boundary may be approximated by linearization. We illustrate this with the classical model of water waves. The problem is to find a potential function φ ($\nabla\varphi$ is the velocity) satisfying in the water region:

$$\Delta\varphi = 0 \quad \text{if} \quad -b(x) < y < f(x,t) \qquad (x = (x_1, x_2))$$

where $y = -b(x)$ is the fixed bottom, $y = f(x,t)$ is the free surface of the water, and the boundary conditions:

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad y = -b(x) ,$$

and

$$\frac{\partial \varphi}{\partial n} = V_n , \quad \frac{\partial \varphi}{\partial t} + \frac{1}{2} |\nabla \varphi|^2 + gy = 0 \quad \text{on the free boundary } y = f(x,t) ;$$

here V_n is the velocity of the free boundary and $\partial/\partial n$ is the derivative in the normal direction.

If $|\frac{\partial f(x,t)}{\partial x}| \ll 1$, we linearize about the equilibrium solution with $\varphi = 0$ and free boundary $y = 0$ by taking

$$f(x,t) = \varepsilon \eta(x,t) , \quad \varphi = \varepsilon u ,$$

and dropping higher order terms. We obtain

$$\Delta u = 0 \quad \text{if} \quad y < 0 , \tag{1.16}$$

with

$$\frac{\partial u}{\partial y} = \frac{\partial \eta}{\partial t} , \quad \frac{\partial u}{\partial t} + g\eta = 0 \quad \text{on} \quad y = 0 .$$

Eliminating η results in the boundary condition

$$\frac{\partial^2 u}{\partial t^2} + g \frac{\partial u}{\partial y} = 0 \quad \text{on} \quad y = 0 . \tag{1.17}$$

The system (1.16), (1.17) supplemented by initial and boundary conditions is the well known *linearized water wave problem*.

Percolation in a Sand-bank Consider next percolation of sea water in gently sloping sand-bank $-b < y < \varepsilon g(x)$, $\varepsilon > 0$ and $\varepsilon \ll 1$; we take x to be 1-dimensional. Denote by p the pressure of the sea water in the sand-bank. By Darcy's law (taking the gravity constant =1)

$$p(x,y,t) = -y + \varepsilon u(x,y,t)$$

in the wet part of the sand-bank, where we have normalized so that $-\nabla p$ is the velocity of the sea water. On the free boundary Γ which is the boundary of the dry portion of the sand-bank, $p = 0$ and, by the continuity equation, also

$$-\varepsilon \frac{\partial u}{\partial n} = V_n$$

where V_n is the velocity of the free boundary. If we denote the free boundary by $y = \varepsilon h(x,t)$, then to lower order, on the free boundary,

$$\frac{\partial u}{\partial y} = -\frac{\partial h}{\partial t} \tag{1.18}$$

and

$$u = h \qquad (\text{since} \quad p = 0). \tag{1.19}$$

On the wet part $p = 0$, so $u = g(x)$ and, for outflow, $\partial u/\partial y \leq 0$. By linearization about $y = 0$ we then get:

$$\Delta u = 0 \quad \text{if} \quad -b < y < 0 \,, \tag{1.20}$$

$$u \leq g \,, \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \leq 0 \,, \quad (u - g)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\right) = 0 \quad \text{on} \quad y = 0 \tag{1.21}$$

with

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad y = -b \,. \tag{1.22}$$

The dry part corresponds to $u(x, 0, t) < g(x)$ and the wet part corresponds to $u(x, 0, t) = g(x)$.

The motivation for this problem is illustrated in Figure 1.2 where a plant requires cooling water for its operation.

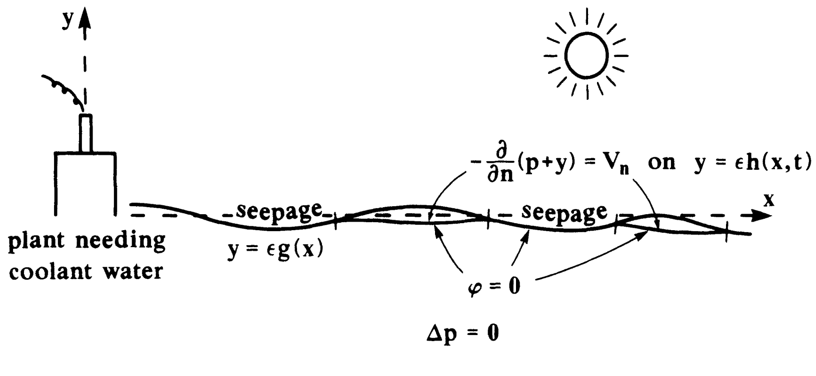

FIGURE 1.2

The system (1.20)–(1.22) supplemented by initial and boundary conditions was formulated in [6]. Existence, uniqueness and properties of the solution have been established in [7].