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Fibrewise Homotopy Theory

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Preface

Topology occupies a central position in the mathematics of today. One of the most useful ideas to be introduced in the past sixty years is the concept of fibre bundle, which provides an appropriate framework for studying differential geometry and much else. Fibre bundles are examples of the kind of structures studied in fibrewise topology.

Just as homotopy theory arises from topology, so fibrewise homotopy theory arises from fibrewise topology. In this monograph we provide an overview of fibrewise homotopy theory as it stands at present. It is hoped that this may stimulate further research. The literature on the subject is already quite extensive but clearly there is a great deal more to be done.

Efforts have been made to develop general theories of which ordinary homotopy theory, equivariant homotopy theory, fibrewise homotopy theory and so forth will be special cases. For example, Baues [7] and, more recently, Dwyer and Spalinski [53], have presented such general theories, derived from an earlier theory of Quillen, but none of these seem to provide quite the right framework for our purposes. We have preferred, in this monograph, to develop fibrewise homotopy theory more or less *ab initio*, assuming only a basic knowledge of ordinary homotopy theory, at least in the early sections, but our aim has been to keep the exposition reasonably self-contained.

Fibrewise homotopy theory has attracted a good deal of research interest in recent years, and it seemed to us that the time was ripe for an expository survey. The subject is at a less mature stage than equivariant homotopy theory, to which it is closely related, but even so the wealth of material available makes it impossible to cover everything. For example, we do not deal with the recent work [51] of Dror Farjoun on the localization of fibrations.

This monograph is divided into two parts. The first provides a survey of fibrewise homotopy theory, beginning with an outline of the basic theory and proceeding to a selection of applications and more specialized topics. The second part is concerned with the stable theory; the emphasis is on theory appropriate for geometric applications, and it is hoped that the account will be accessible to readers who may not already be experts in the classical stable theory. Part II does assume a certain familiarity with the basic ideas from Part I, but is written in such a way that the reader interested mainly in the stable theory should be able to begin with Part II and refer back to

Part I as necessary. More details on the contents of specific sections can be found in the Introductions to the two parts. Cross-referencing within each part is by section number. We have not attempted a complete bibliography of publications related to fibrewise homotopy theory; those which are cited in either Part I or Part II are listed at the end of Part II. Similarly, the index at the end of the book covers both parts.

Certain sections are based on previously published work, and where appropriate this is mentioned in the text. We are grateful to the publishers in question for permission to include this material.

Our thinking on fibrewise homotopy theory has been influenced by the work of many colleagues, but we owe a special debt to those with whom we have collaborated on joint papers (both published and unpublished). We are grateful to our co-authors for sharing their insight with us. MCC would like to record, in particular, his thanks to Andrew Cook, Karlheinz Knapp and Wilson Sutherland.

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Part I. A Survey of Fibrewise Homotopy Theory

Introduction

The basic ideas of fibrewise homotopy theory seem to have occurred in the late 1960s to several people independently. Thus J.C. Becker [8], J.F. McClendon [106], L. Smith [125] and I.M. James [78] all made use of the theory in work published in 1969 or 1970, while several others, such as L. Hodgkin and J.-P. Meyer, were also well aware of its possibilities. I.M. James first published a systematic account of the basic theory in 1985 [85] but this was largely based on much earlier work [78] put aside when he became aware that so many others were thinking on the same lines. Five years later, after further research, he returned to the subject in [86]. Although the present exposition is to some extent based on these earlier accounts it mainly consists of new material.

There is some truth in the observation that once the correct definitions have been formulated any well-organized and methodical account of the relevant homotopy theory, such as [44], can be converted to fibrewise homotopy theory by writing in the word 'fibrewise' wherever it makes sense and adjusting formulae accordingly. Yet even at the most elementary level it is necessary to exercise care and not jump to conclusions, just as it is in the case of equivariant homotopy theory. One might hope that some completely routine way may be found of producing fibrewise versions of results in ordinary homotopy theory but it would be an exaggeration to say that this is possible at present, although Heller [74] suggests that a way may be found.

On the question of terminology, we find it best on the whole to try and use the term *fibrewise* throughout. For example we now prefer the term *fibrewise pointed space* to the alternatives such as *sectioned space, ex-space*, etc. One reason is that fibrewise corresponds closely to the French *fibré* and the German *faserweise*. However, excessive repetition of the term fibrewise may seem monotonous and so we make the convention that it governs the words which come after it so that the expression *fibrewise compact Hausdorff space*, for example, means *fibrewise compact, fibrewise Hausdorff, fibrewise space*. Of course, the time may come when it will be possible to leave out the term fibrewise, just as one does in vector bundle theory, and to simplify the notations accordingly. However, experience suggests that to do so at the present time is liable to cause confusion.

In the exposition which follows we assume that the reader is familiar with the basic notions of ordinary homotopy theory, as set out in [44], for example. Routine fibrewise versions of proofs of well-known results in the ordinary theory are generally omitted. Otherwise, and except for certain examples, the exposition is fairly self-contained.

The text is divided into three chapters, each consisting of a number of sections. Chapter 1 is concerned with the category of fibrewise spaces and fibrewise maps, classified by fibrewise homotopy. As we shall see, it is not always obvious what is the most appropriate fibrewise version of a concept in ordinary homotopy theory. Chapter 2 is concerned with the category of fibrewise pointed spaces and fibrewise pointed maps, classified by fibrewise pointed homotopy. In the ordinary theory not a great deal of attention is usually paid to the difference between the pointed theory and the non-pointed theory but in the fibrewise version the difference is vital. More specialized topics are considered in Chapter 3. Several of the sections are closely modelled on material which has appeared elsewhere: Sections 17 and 20 are edited versions of [92] and [63], respectively; Sections 19 and 21 are based on [90] and [89], respectively, and Sections 22 and 23 have been extracted from [31].

Chapter 1. An Introduction to Fibrewise Homotopy Theory

1 Fibrewise spaces

Basic notions

Let us work over a (topological) base space B. A *fibrewise space* over B consists of a space X together with a map $p : X \to B$, called the *projection*. Usually X alone is sufficient notation. We regard any subspace of X as a fibrewise space over B by restricting the projection. When p is a fibration we describe X as *fibrant*.

We regard B as a fibrewise space over itself using the identity as the projection. We regard the topological product $B \times T$, for any space T, as a fibrewise space over B using the second projection.

Let X be a fibrewise space over B. For each point b of B the *fibre* over b is the subset $X_b = p^{-1}b$ of X; fibres may be empty since we do not require p to be surjective. Also for each subspace B' of B we regard $X_{B'} = p^{-1}B'$ as a fibrewise space over B' with projection p' determined by p.

Fibrewise spaces over B constitute a category with the following definition of morphism. Let X and Y be fibrewise spaces over B with projections p and q, respectively. A *fibrewise map* $\phi : X \to Y$ is a map in the ordinary sense such that $q \circ \phi = p$, in other words such that $\phi X_b \subseteq Y_b$ for each point b of B. If $\phi : X \to Y$ is a fibrewise map over B then the restriction $\phi_{B'} : X_{B'} \to Y_{B'}$ is a fibrewise map over B' for each subspace B' of B. Thus a functor is defined from the category of fibrewise spaces over B to the category of fibrewise spaces over B'.

Equivalences in the category of fibrewise spaces over B are called *fibrewise topological equivalences* or *fibrewise homeomorphisms*. If ϕ, as above, is a fibrewise topological equivalence over B then $\phi_{B'}$ is a fibrewise topological equivalence over B' for each subspace B' of B. In particular ϕ_b is a topological equivalence for each point b of B. However, this necessary condition for a fibrewise topological equivalence is obviously not sufficient. To see this take $Y = B$ to be a non-discrete space and take X to be the same set with the discrete topology and the identity as projection; the identity function has no continuous inverse.

A fibrewise map $\phi : X \to Y$ is said to be *fibrewise constant* if $\phi = t \circ p$ for some section $t : B \to Y$. The same example as in the previous paragraph shows that a fibrewise map may be constant on each fibre but not fibrewise constant.

Fibrewise product and coproduct

Given an indexed family $\{X_j\}$ of fibrewise spaces over B the *fibrewise product* $\prod_B X_j$ is defined as a fibrewise space over B, and comes equipped with a family of fibrewise projections

$$\pi_j : \prod_B X_j \to X_j.$$

The fibres of the fibrewise product are just the products of the corresponding fibres of the factors. The fibrewise product is characterized by the following Cartesian property: for each fibrewise space X over B the fibrewise maps

$$\phi : X \to \prod_B X_j$$

correspond precisely to the families of fibrewise maps $\{\phi_j\}$, where

$$\phi_j = \pi_j \circ \phi : X \to X_j.$$

For example if $X_j = X$ for each index j the *diagonal*

$$\Delta : X \to \prod_B X$$

is defined so that $\pi_j \circ \Delta = 1_X$ for each j.

If $\{X_j\}$ is as before the *fibrewise coproduct* $\coprod_B X_j$ is also defined, as a fibrewise space over B, and comes equipped with a family of fibrewise insertions

$$\sigma_j : X_j \to \coprod_B X_j.$$

The fibres of the fibrewise coproduct are just the coproducts of the corresponding fibres of the summands. The fibrewise coproduct is characterized by the following cocartesian property: for each fibrewise space X over B the fibrewise maps

$$\psi : \coprod_B X_j \to X$$

correspond precisely to the families of fibrewise maps $\{\psi_j\}$, where

$$\psi_j = \psi \circ \sigma_j : X_j \to X.$$

For example if $X_j = X$ for each index j the *codiagonal*

$$\nabla : \coprod_B X \to X$$

is defined so that $\nabla \circ \sigma_j = 1_X$ for each j.

The notations $X \times_B Y$ and $X \sqcup_B Y$ are used for the fibrewise product and fibrewise coproduct in the case of a family $\{X, Y\}$ of two fibrewise spaces, and similarly for finite families generally. When $X = Y$ the switching maps

$$X \times_B X \to X \times_B X, \qquad X \sqcup_B X \to X \sqcup_B X$$

are defined with components (π_2, π_1) and (σ_2, σ_1), respectively.

Given a map $\alpha : B' \to B$, for any space B', we can regard B' as a fibrewise space over B. For each fibrewise space X over B we denote by $\alpha^* X$ the fibrewise product $X \times_B B'$, regarded as a fibrewise space over B' using the second projection, and similarly for fibrewise maps. Thus α^* constitutes a functor from the category of fibrewise spaces over B to the category of fibrewise spaces over B'. When B' is a subspace of B and α the inclusion this is equivalent to the restriction functor described earlier.

By a *fibrewise topology*, on a fibrewise set X over B, we mean any topology on X such that the projection p is continuous. By a *fibrewise basis*, for a fibrewise topology, we mean a collection $\mathcal{U}$ of subsets of X which forms a basis for a topology after augmentation by the topology induced by p. In other words, the open sets of X are the unions of intersections of members of $\mathcal{U}$ and sets of the form X_W, where W is open in B. For example, consider the product $B \times T$, where T is a space. A fibrewise basis for the fibrewise topology is given by the collection of products $B \times U$, where U runs through the open sets of T.

The term *fibrewise sub-basis* is used in a similar sense or we may, on occasion, say that the fibrewise topology is generated by a family of subsets, meaning that finite intersections of members of the family form a fibrewise basis.

Note that in checking the continuity of fibrewise functions, where the fibrewise topology of the codomain is generated in this way, it is sufficient to verify that the preimages of fibrewise subbasic open sets are open.

Fibre bundles

A fibrewise space X over B is said to be *trivial* if X is fibrewise homeomorphic to $B \times T$ for some space T, and then a fibrewise homeomorphism $\phi : X \to B \times T$ is called a *trivialization* of X. A fibrewise space X over B is said to be *locally trivial* if there exists an open covering of B such that X_V is trivial over V for each member V of the covering. A locally trivial fibrewise space is the simplest form of fibre bundle or bundle of spaces. As Dold [45] has shown, the theory of fibre bundles is improved if it is confined to the class of numerable bundles, i.e. bundles which are trivial over every member of some numerable covering of the base. Derwent [40] and tom Dieck [42] have pointed out that such a covering may be taken to be countable, thus facilitating inductive arguments.

A more sophisticated form of the notion of fibre bundle involves a topological group G, the structural group. A principal G-bundle over the base space B is a locally trivial fibrewise space P over B on which G acts freely. Moreover, the action is fibre-preserving, so that each of the fibres is homeomorphic to G. Such a principal G-bundle P over B determines a functor

$P_\#$ from the category of G-spaces to the category of fibre bundles over B. Specifically $P_\#$ transforms each G-space A into the associated bundle $P \times_G A$ with fibre A, and similarly with G-maps. We refer to $P_\#$ as the *associated bundle functor.*

The theory of fibre bundles is dealt with in the standard textbooks such as Steenrod [128] or Bredon [19], where a large variety of examples are discussed. Some of these will be appearing in the course of our work.

From our point of view it is only natural to proceed a stage further and develop a fibrewise version of the theory of fibre bundles, as in [95]. Thus let X and T be fibrewise spaces over B. By a *fibrewise fibre bundle* over X, with fibrewise fibre T, we mean a fibrewise space E together with a fibrewise map $p : E \to X$ which is locally fibrewise trivial, in the sense that there exists a covering of X such that E_V is fibrewise homeomorphic to $V \times_B T$, over B, for each member V of the covering. This is the simplest form of the definition, but of course there is a more sophisticated form, involving a fibrewise structural group. Details are given in Section 8 below.

Classes of fibrewise spaces

There are various classes of fibrewise spaces which will appear in the work we shall be doing later, for example, the class of fibrewise open spaces, where the projection is open. To be of any interest to us such a class must be invariant, so that a fibrewise space which is fibrewise homeomorphic to a member of the class is also a member of the class. It must also be natural, in the sense that pull-backs of a member are also members. Furthermore, fibrewise products of members are also members, at least finite fibrewise products. Fibre bundles are such a class.

In fibrewise topology the existence of local sections is a condition of some importance, but more usually it is the existence of local slices which is required.

Definition 1.1 The fibrewise space X over B is *locally sliceable* if for each point b of B and each point x of X_b there exists a neighbourhood W of b and a section $s : W \to X_W$ such that $s(b) = x$.

The condition implies that p is open since if U is a neighbourhood of x in X then $s^{-1}(X_W \cap U) \subseteq pU$ is a neighbourhood of b in W. In other words, locally sliceable fibrewise spaces are fibrewise open.

There are fibrewise versions of all the usual separation conditions of topology, in fact the number of different fibrewise separation conditions which can reasonably be defined is quite large. For our purposes, however, only two or three are of real significance.

Definition 1.2 The fibrewise space X over B is *fibrewise Hausdorff* if the diagonal embedding

$$\Delta : X \to X \times_B X$$

is closed.

Equivalently, for each point b of B and each pair x, x' of distinct points of X_b there exist disjoint neighbourhoods of x, x' in X.

Subspaces of fibrewise Hausdorff spaces are fibrewise Hausdorff. The following two properties of fibrewise Hausdorff spaces are worth mentioning.

Proposition 1.3 *Let $\phi : X \to Y$ be a fibrewise map, where X and Y are fibrewise spaces over B. If Y is fibrewise Hausdorff the fibrewise graph of ϕ is closed in $X \times_B Y$.*

Proposition 1.4 *Let $\phi, \psi : X \to Y$ be fibrewise maps, where X and Y are fibrewise spaces over B. If Y is fibrewise Hausdorff the coincidence set $K(\phi, \psi)$ of ϕ and ψ is closed in X.*

These results follow easily from the definition.

From the viewpoint of fibrewise topology it seems natural to revise some of the terminology of ordinary topology. For example

Definition 1.5 The fibrewise space X over B is *fibrewise discrete* if the projection p is a local homeomorphism.

Clearly, fibrewise discrete spaces are locally sliceable and hence fibrewise open. An attractive characterization of this class of fibrewise spaces is given by

Proposition 1.6 *Let X be a fibrewise space over B. Then X is fibrewise discrete if and only if* (i) *X is fibrewise open and* (ii) *the diagonal embedding*

$$\Delta : X \to X \times_B X$$

is open.

Corollary 1.7 *Let $\phi : X \to Y$ be a fibrewise map, where X is fibrewise open and Y is fibrewise discrete over B. Then the fibrewise graph*

$$\Gamma : X \to X \times_B Y$$

of ϕ is an open embedding.

Corollary 1.8 *Let $\phi, \psi : X \to Y$ be fibrewise maps, where X and Y are fibrewise spaces over B. If Y is fibrewise discrete the coincidence set $K(\phi, \psi)$ of ϕ and ψ is open in X.*

Another fibrewise separation condition we shall need is as follows.

Definition 1.9 The fibrewise space X over B is *fibrewise regular* if for each point b of B, each point x of X_b and each neighbourhood V of x in X there exists a neighbourhood W of b in B and a neighbourhood U of x in X_W such that the closure $X_W \cap \bar{U}$ of U in X_W is contained in V.

When the fibrewise topology of X is given in terms of a fibrewise sub-basis it is sufficient if the condition for fibrewise regularity is satisfied for fibrewise subbasic neighbourhoods V. Subspaces of fibrewise regular spaces are also fibrewise regular, as can easily be shown.

Fibrewise open means that the projection is open, fibrewise closed that the projection is closed. Because fibrewise products of fibrewise closed spaces are not, in general, fibrewise closed, the class of fibrewise closed spaces is only of minor importance. A stronger condition is needed, as in

Definition 1.10 The fibrewise space X over B is *fibrewise compact* if the projection is proper.

In other words X is fibrewise compact if X is fibrewise closed and every fibre of X is compact. One can also characterize the condition in terms of coverings, as follows.

Proposition 1.11 *The fibrewise space X over B is fibrewise compact if and only if for each point b of B and each covering $\mathcal{U}$ of X_b by open sets of X there exists a neighbourhood W of b in B such that a finite subfamily of $\mathcal{U}$ covers X_W.*

Proposition 1.12 *Let X be fibrewise compact over B. Suppose that X is fibrewise discrete. Then $X \to B$ is a finite covering space.*

For consider a point $b \in B$. Choose for each $x \in X_b$ an open neighbourhood U_x in X such that $p(U_x)$ is open in B and the restriction of p is a homeomorphism $U_x \to p(U_x)$. Since the intersection of U_x with the fibre X_b is precisely $\{x\}$, it follows from Proposition 1.11 that X_b is finite and that there is an open neighbourhood W of b in B such that $X_W \subseteq \bigcup_x U_x$. Let V be the open subset

$$V = W \cap \bigcap_x p(U_x) \subseteq B.$$

Then $X_V \to V$ is trivial.

The images of fibrewise compact spaces under fibrewise maps are also fibrewise compact. This follows at once from the definition; with a little more effort we obtain the useful

Proposition 1.13 *Let $\phi : X \to Y$ be a fibrewise map, where X is fibrewise compact and Y is fibrewise Hausdorff. Then ϕ is proper.*

Proposition 1.14 *Let X be fibrewise regular over B and let K be a fibrewise compact subset of X. Let b be a point of B and let V be a neighbourhood of K_b in X. Then there exists a neighbourhood W of b in B and a neighbourhood U of K_W in X_W such that the closure $X_W \cap \bar{U}$ of U in X_W is contained in V.*

There is just one more class of fibrewise spaces we need to consider here.

Definition 1.15 The fibrewise space X over B is *fibrewise locally compact* if for each point b of B and each point x of X_b there exists a neighbourhood W of b in B and a neighbourhood U of x in X_W such that the closure $X_W \cap \bar{U}$ of U in X_W is fibrewise compact over W.

It is easy to see that fibrewise compact spaces are fibrewise locally compact, also that closed subspaces of fibrewise locally compact spaces are fibrewise locally compact. We conclude with two results which are not quite so obvious; proofs may be found in Section 3 of [86]. Recall that we make the convention that the term 'fibrewise' governs everything that follows it. For example 'fibrewise locally compact Hausdorff space' means a fibrewise space which is both fibrewise locally compact and fibrewise Hausdorff.

Proposition 1.16 *Let X be fibrewise locally compact Hausdorff over B. Then X is fibrewise regular.*

Proposition 1.17 *Let X be fibrewise locally compact regular over B. Then for each point b of B, each compact subset C of X_b, and each neighbourhood V of C in X, there exists a neighbourhood W of b in B and a neighbourhood U of C in X_W such that the closure $X_W \cap \bar{U}$ of U in X_W is fibrewise compact over W and contained in V.*

Fibrewise quotients

By a *fibrewise quotient map* we mean a fibrewise map which is a quotient map in the ordinary sense. Fibrewise products of fibrewise quotient maps are not necessarily fibrewise quotient maps. We prove

Proposition 1.18 *Let $\phi : X \to Y$ be a fibrewise quotient map, where X and Y are fibrewise spaces over B. Then the fibrewise product*

$$\phi \times 1 : X \times_B T \to Y \times_B T$$

is a fibrewise quotient map, for all fibrewise locally compact regular T.

For let $U \subseteq X \times_B T$ be open and saturated with respect to $\psi = \phi \times 1$. We have to show that ψU is open in $Y \times_B T$. So let $(y, t) \in \psi U$, where $y \in Y_b$,

$t \in T_b$, $b \in B$, and pick $x \in \phi^{-1}(y) \subseteq X_b$. We have $(x,t) \in U$, since U is saturated. Consider the subset N of T_b given by

$$\{x\} \times N = (\{x\} \times T_b) \cap U.$$

Now N is open in T_b, since U is open in $X \times_B T$, and so $N = M \cap T_b$, where M is open in T. Since T is fibrewise locally compact there exists, by Proposition 1.17, a neighbourhood $K \subseteq M$ of t in T_W such that K is fibrewise compact over W. Consider the subset

$$V = \{\xi \in X_W \mid \{\xi\} \times_W K \subseteq U\}$$

of X_W. We have $(y,t) \in \phi V \times_W K \subseteq \psi U$. So to prove that ψU is a neighbourhood of (y,t) in $Y \times_B T$ it is sufficient to prove that ϕV is a neighbourhood of y in Y.

In fact V is open in X. For let $\xi \in V$ so that $\{\xi\} \times_W K_\beta \subseteq U$, where $\beta = p(\xi)$ and $p : X \to B$. Since K is fibrewise compact over W the projection

$$X_W \times_W K \to X_W \times_W W \to X_W$$

is closed. Since U is a neighbourhood of the inverse image $\{\xi\} \times K_\beta$ of ξ under the projection there exists a neighbourhood $W' \subseteq W$ of β and a neighbourhood V' of ξ in $X_{W'}$ such that $V' \times_{W'} K_{W'} \subseteq U$. This implies that $V' \subseteq V$, by the definition of V, and so V is open.

Moreover, V is saturated. For $V \subseteq \phi^{-1}\phi V$, as always. Also

$$\phi^{-1}\phi V \times_W K = \psi^{-1}\psi(V \times_W K) \subseteq \psi^{-1}\psi U = U.$$

Therefore $\phi^{-1}\phi V \subseteq V$, by the definition of V, and so $\phi^{-1}\phi V = V$. Thus V is saturated, as well as open, and so ϕV is open. Since $y \in \phi V$ this completes the proof.

Given a fibrewise space X over B a *fibrewise equivalence relation* on X is given by a subset R of the fibrewise product $X \times_B X$. We refer to the fibrewise set X/R of equivalence classes, with the quotient topology, as the *fibrewise quotient space.* Of course, fibrewise maps $X/R \to Z$, for any fibrewise space Z, correspond precisely to invariant fibrewise maps $X \to Z$.

We describe a fibrewise map $\phi : (X, A) \to (X', A')$ as a *fibrewise relative homeomorphism* if (i) A is closed in X, (ii) ϕ maps $X - A$ bijectively onto $X' - A'$, and (iii) X' is a fibrewise quotient space of X under ϕ.

In general there is no simple condition at the level of X which implies that X/R is fibrewise Hausdorff. Suppose, however, that $R = (\phi \times \phi)^{-1}\Delta Z$ for some fibrewise map $\phi : X \to Z$, where Z is fibrewise Hausdorff. Then the induced fibrewise map $X/R \to Z$ is injective and so X/R is fibrewise Hausdorff.

Consider a space D and a closed subspace E of D. For any fibrewise space X over B let $\Phi_B(X)$ denote the push-out of the cotriad

$$X \times D \xleftarrow{\supset} X \times E \xrightarrow{\pi_2} E,$$

and similarly for fibrewise maps. Thus an endofunctor Φ_B of our category is defined. It can be shown, as in [86], that $\Phi_B(X)$ is fibrewise Hausdorff whenever X is fibrewise Hausdorff. For $(D, E) = (I, \{0\})$, where $I = [0, 1] \subseteq \mathbb{R}$, the endofunctor is known as the *fibrewise cone* and denoted by C_B. When $(D, E) = (I, \{0, 1\})$ the endofunctor is known as the *fibrewise suspension* and denoted by Σ_B. For example $\Sigma_B(X) = B \times I$ when $X = B$, and $\Sigma_B(X) = B \times \dot{I}$ when $X = \emptyset$. Note that the associated bundle functor $P_\#$ discussed earlier, from the category of G-spaces to the category of fibrewise spaces, transforms the equivariant cone into the fibrewise cone and the equivariant suspension into the fibrewise suspension. For example, taking G to be the orthogonal group $O(n)$, the fibrewise cone of an $(n-1)$-sphere bundle is the associated n-ball bundle, and the fibrewise suspension is the associated n-sphere-bundle.

More generally let X_i $(i = 0, 1)$ be a fibrewise space. Consider the fibrewise equivalence relation on the coproduct

$$X_0 \sqcup (X_0 \times I \times X_1) \sqcup X_1$$

which identifies (x_0, t, x_1), with x_t whenever $t = 0$ or 1. The fibrewise set $X_0 *_B X_1$ of equivalence classes, with the quotient topology, is called the *fibrewise join* of X_0 and X_1. For example, if X_t is the sphere-bundle associated with E_t, where E_t is a euclidean bundle over B, then $X_0 *_B X_1$ is the sphere-bundle associated with the Whitney sum $E_0 \oplus E_1$. When $X_0 = S^{n-1} \times B$ and $X_1 = X$ we may identify $X_0 *_B X_1$ with the n-fold fibrewise suspension $\Sigma_B^n(X)$ of the fibrewise space X.

It should be noted that the fibrewise join is not in general associative, with the quotient topology. However if, following Milnor [113], we replace this by the coarsest topology which makes the coordinate functions

$$\begin{aligned} t &: X_0 *_B X_1 \to B \times I, \\ x_0 &: t^{-1}(B \times [0, 1)) \to X_0, \\ x_1 &: t^{-1}(B \times (0, 1]) \to X_1 \end{aligned}$$

continuous then associativity holds without restriction. Furthermore, the topologies coincide when X_0 and X_1 are fibrewise compact Hausdorff.

Fibrewise mapping-spaces

Finally, let us turn to the problem of constructing a right adjoint to the fibrewise product. One has to impose a topology with the necessary properties on the fibrewise set

$$\operatorname{map}_B(X, Z) = \coprod_{b \in B} \operatorname{map}(X_b, Z_b),$$

where X and Z are fibrewise spaces over B. Although this can be done in general, as we shall see later, the case when $X = B \times T$, for some space T, admits of simpler treatment. In fact maps of $\{b\} \times T$ into Z_b can be regarded as maps of T into Z, in the obvious way, and so $\mathrm{map}_B(B \times T, Z)$ can be topologized as a subspace of $\mathrm{map}(T, Z)$, with the compact-open topology. It is easy to check that for any fibrewise space Y over B a fibrewise map

$$Y \times T = (B \times T) \times_B Y \to Z$$

determines a fibrewise map

$$Y \to \mathrm{map}_B(B \times T, Z),$$

through the standard formula, and that the converse holds when T is compact Hausdorff. In fact this special case is sufficient for the great majority of situations where fibrewise mapping-spaces are used in what follows.

Some examples

The reader may wish to treat the following examples, related to the text of this section, as exercises.

Example 1.19. Let $\phi : X \to Y$ be an open and closed fibrewise surjection where X and Y are fibrewise spaces over B. Let $\lambda : X \to \mathbb{R}$ be a continuous real-valued function which is fibrewise bounded above, in the sense that λ is bounded above on each fibre of X. Then $\mu : Y \to \mathbb{R}$ is continuous, where

$$\mu(\eta) = \sup_{\xi \in \phi^{-1}(\eta)} \lambda(\xi).$$

Example 1.20. Let $\phi : X \to Y$ be a fibrewise function, where X and Y are fibrewise spaces over B. Suppose that X is fibrewise open and that the product

$$id \times \phi : X \times_B X \to X \times_B Y$$

is open. Then ϕ itself is open.

Example 1.21. Let X be a closed subspace of $B \times \mathbb{R}^n$, $(n \geq 0)$, regarded as a fibrewise space over B under the first projection. Then X is fibrewise compact if X is fibrewise bounded, in the sense that there exists a continuous real-valued function $\lambda : B \to \mathbb{R}$ such that X_b is bounded by $\lambda(b)$ for each point b of B.

Example 1.22. Let $\phi : X \to Y$ be a fibrewise function, where X and Y are fibrewise spaces over B. Then, if X is fibrewise compact and the product

$$id \times \phi : X \times_B X \to X \times_B Y$$

is proper, ϕ is proper.