Singularity Theory I

Springer-Verlag Berlin Heidelberg GmbH

V. I. Arnold V. V. Goryunov
O. V. Lyashko V. A. Vasil'ev

Singularity Theory I

Springer

Title of the Russian edition:
Itogi nauki i tekhniki, Sovremennye problemy matematiki,
Fundamental'nye napravleniya, Vol. 6, Dinamicheskie sistemy 6
Publisher VINITI, Moscow 1988

Second Printing 1998 of the First Edition 1993, which was originally
published as Dynamical Systems VI,
Volume 6 of the Encyclopaedia of Mathematical Sciences.

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Singularity theory / by V. I. Arnol'd ... - Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hongkong ; London ; Mailand ; Paris ; Santa Clara ; Singapur ; Tokio : Springer
1. - 1. ed., 2. printing. - 1998
(Encyclopaedia of mathematical sciences ; Vol. 6)
ISBN 978-3-540-63711-0 ISBN 978-3-642-58009-3 (eBook)
DOI 10.1007/978-3-642-58009-3

Mathematics Subject Classification (1991):
Primary 58C27, Secondary 14B05, 14E15, 32S05, 32Sxx, 58C28

ISBN 978-3-540-63711-0

Originally published by Springer-Verlag Berlin Heidelberg New York in 1998

SPIN: 10648999
46/3143-5 4 3 2 1 0 – Printed on acid-free paper.

List of Editors, Authors and Translators

Editor-in-Chief

R.V. Gamkrelidze, Russian Academy of Sciences, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow, Institute for Scientific Information (VINITI), ul. Usievicha 20 a, 125219 Moscow, Russia; e-mail: gam@ipsun.ras.ru

Consulting Editor

V. I. Arnold, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow, Russia, e-mail: arnold@mi.ras.ru; CEREMADE, Université Paris 9 – Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16-e, France, e-mail: arnold@ceremade.dauphine.fr

Authors

V. I. Arnold, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow, Russia, e-mail: arnold@mi.ras.ru; CEREMADE, Université Paris 9 – Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16-e, France, e-mail: arnold@ceremade.dauphine.fr

V. V. Goryunov, Moscow Aviation Institute, Volokolamskoe sh. 4, 125871 Moscow, Russia; Division of Pure Mathematics, The University of Liverpool, Liverpool L69 3BX,UK, e-mail: goryunov@liv.ac.uk

O. V. Lyashko, Airforce Engineering Academy, Leningradskij pr. 40, 125167 Moscow, Russia; e-mail: ovl@aha.ru

V. A. Vasil'ev, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow, Russia; e-mail: vassil@vassil.mccme.rssi.ru

Translator

A. Iacob, Mathematical Reviews, 416 Fourth Street, P.O. Box 8604, Ann Arbor, MI 48107, USA

Singularities
Local and Global Theory

V.I. Arnol'd, V.A. Vasil'ev,
V.V. Goryunov, O.V. Lyashko

Translated from the Russian
by A. Iacob

Contents

Foreword

Napoleon condemned Laplace for the attempt to "introduce the spirit of infinitesimals in government" and removed him from the post of minister.

In this two-volume survey,[1] an exposition is given of the foundations of the part of the analysis of infinitesimals that is necessary for the deliberate control of dynamical systems, for their optimization, and for understanding the behavior of complex systems that depend on several parameters.

The theory of singularities of smooth maps is an apparatus for the study of abrupt, jump-like phenomena – bifurcations, perestroikas (restructurings), catastrophes, metamorphoses – which occur in systems depending on parameters when the parameters vary in a smooth manner.

Although the applications to the theory of dynamical systems do not exhaust by far all the potential capabilities of the theory of singularities (it also has applications in geometric and physical optics, hydrodynamics, quantum mechanics, crystallography, chemistry, acoustics, sinergetics, the theory of radio-wave propagation, cosmology, algebraic geometry, differential topology, and so forth), the fundamental role that the theory of singularities plays in the investigation of bifurcations of stationary and periodic regimes justifies the inclusion of this two-volume book in the series "Dynamical Systems".

In writing the survey the authors had in mind a student-reader, mathematician or physicist, who wishes to learn the modern mathematical apparatus of local mathematical analysis as an instrument for applied studies, or a specialist in the respective applied domain seeking for the needed mathematical tools and reference information. Accordingly, we replaced proofs by references to the sources where they can be found, focussing on the methods, ideas, and results, rather than the technical details of the proofs. In doing so we counted on a reader prepared to accept many details on trust, or prefering to reconstruct the proofs by himself. A thorough exposition in any of the formalized mathematical languages (be it "$\forall\varepsilon\ \exists\delta$", "Ext-Tor", or "GO TO") would have required a many times greater volume.

The first two chapters of this volume are devoted to one of the most advanced parts of the theory of singularities – the study of degeneracies of critical points of functions.

In Chapter 1 we acquaint the reader with the basic notions of the theory of singularities of smooth maps, give the initial segment of the classification of smooth functions and present the technique of reducing functions to normal forms.

In Chapter 2 we consider topological and algebro-geometric aspects of the theory of critical points of functions. Here we discuss the basic concepts of the local Picard-Lefschetz theory, that is, the discipline of branching of cycles and

[1] The second part of the survey, "Singularity Theory II. Applications", will appear as Vol. 39 of the present series (Dynamical Systems VIII).

integrals depending on parameters. A detailed study is made of the main object of this theory – the bundle of vanishing cohomology (i.e., of branching integration contours) connected with a critical point and, in particular, the base over which this bundle is defined, the complement of the discriminant of the singularity. We also consider the connection between simple singularities of functions and the classification of simple Lie groups, reflection groups, and braid groups.

Among the original results of this chapter let us mention the calculation of the cohomology groups with nonconstant coefficients of the complements of discriminants of one-dimensional singularities and their application to the theory of algorithms, the description of the stable cohomology of the complements of discriminants of arbitrary singularities, theorems on stable irreducibility of the strata of a discriminant, the noncoincidence of the dimensions of the complex and real $\mu = \text{const}$ strata of real singularities.

In Chapter 3 we give an exposition of the general theory of equivalence of maps.

Mathematical and physical problems arising in real situations lead to the investigation of the properties of maps with respect to a variety of equivalence relations. In analyzing a concrete equivalence relation one has to deal with a number of standard questions: Is the given map stable? Can one regard the map, even locally, as a polynomial, which would considerably simplify calculations? Does the map admit a versal deformation, i.e., can it be included in a family with a finite number of parameters, which contains all small deformations of the map? How much simpler does the classification become when one passes from the rigid differentiable equivalence to the less demanding topological one? For many equivalence relations the answers to these questions look the same. The statements of the corresponding theorems and of sufficient conditions for their applicability constitute the main content of the third chapter.

In the last, fourth chapter, we describe topological characteristics of singular sets of smooth maps: the cohomology classes dual to the sets of critical points and nonregular values; invariants of maps defined by these classes; the structure of the spaces of maps not having singularities of one kind or another. Apparently for the first timc in the literature, we carry out the construction of characteristic classes of foliations with the help of universal complexes of singularities and multisingularities, and also the computation of the fundamental group of the space of functions with singularities no more complicated than x^3 and of the topology of complements of open swallowtails.

For the first time in monograph form, we discuss in this survey the results of S.V. Chmutov on the monodromy group of an isolated singularity in the skew-symmetric case, the theorems of O.V. Lyashko and P. Jaworski on the decomposition of simple and parabolic singularities, the estimates of the index of a polynomial vector field obtained by A.G. Khovanskiĭ, and the results of E.I. Shustin and V.I. Arnol'd on the number of the flattening points that vanish under various degeneracies of algebraic hypersurfaces.

The references within the volume are organized as follows. If the reference lies within the same chapter, we indicate the number of the corresponding section or subsection, as in the table of contents. If the reference is to a different chapter, then we precede the number of the section or subsection by the number of the chapter.

Chapters 1 and 2, with the exception of § 2.6 and subsections 2.1.11, 2.5.11, were written by O.V. Lyashko, Chapter 3 and subsections 2.1.11, 2.5.11 by V.V. Goryunov, Chapter 4 (except for § 4.4) and § 2.6 by V.A. Vasil'ev, and § 4.4 by V.I. Arnol'd.

For the second volume of the survey V.I. Arnol'd and V.A. Vasil'ev each contributed two chapters, and one chapter was written by V.V. Goryunov. The second volume gives a representation of a wide circle of problems that are currently being solved, and contains many new results.

Chapter 1
Critical Points of Functions

One of the most thoroughly studied branches of the theory of singularities is the investigation and classification of degeneracies of critical points of functions. Generic functions have only nondegenerate critical points. More complex singularities vanish under small perturbations, decomposing into nondegenerate ones.

However, in families of functions that depend on several parameters, degenerate critical points may occur in an irremovable manner. For example, the family of functions $x^3 + \lambda x$ has for the value $\lambda = 0$ of the parameter a degenerate critical point; any close one-parameter family has, for a close value of the parameter, a degeneracy of the same kind. As the number of parameters increases, in families of functions there arise degeneracies of ever increasing complexity.

In this chapter we describe the initial segment of the classification of critical points of functions. This classification of the simplest degeneracies of critical points turned out to be closely related to the classification of simple Lie groups, of reflection groups, and of braid groups.

To simplify the exposition, we restrict ourselves mainly to the case of holomorphic functions, diffeomorphisms, and so on. The theory carries over, with practically no modifications, to the case of real smooth functions; some differences arising in the real case will be pointed out. The classification of real critical points is given in subsection 2.8.

§ 1. Invariants of Critical Points

Here we give the basic definitions concerning critical points of functions.

1.1. Degenerate and Nondegenerate Critical Points

Definition. A point is said to be a *critical point* of a smooth function f if at that point the derivative of f is equal to zero.

The value that the function takes at a critical point is called a *critical value*.

Example. The function $f(x) = x^3 - \lambda x$ of the variable $x \in \mathbb{C}$ has for each $\lambda \neq 0$ the two critical points $\pm\sqrt{\lambda/3}$ with respective critical values $\mp\sqrt{4\lambda^3/27}$. For $\lambda = 0$ these two critical points "merge" into a single critical point 0.

The critical points of functions are divided into generic (general-position, or nondegenerate) critical points and degenerate critical points.

V. I. Arnold et al., *Singularity Theory I*

Definition. A critical point is said to be *nondegnerate* (or a *Morse critical point*) if the second differential of the function at that point is a nondegenerate quadratic form.

Example. The function $f(x) = x^3 - \lambda x$ has for $\lambda \neq 0$ the pair of nondegenerate critical points $\pm\sqrt{\lambda/3}$ and for $\lambda = 0$ the degenerate critical point 0 (Fig. 1).

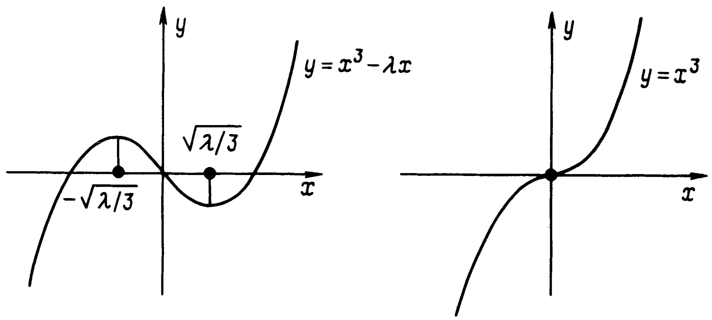

Fig. 1

The degree of degeneracy of the second differential is the simplest indicator of how degenerate a critical point is.

Definition. The *corank* of a critical point of a function is the dimension of the kernel of its second differential at the critical point.

Examples. The corank of any Morse critical point is equal to zero. The corank of the critical point 0 of the function $f = x_1^3 + x_2^2 + \cdots + x_n^2$ is equal to one.

1.2. Equivalence of Critical Points. Let us consider the set $\mathcal{O}_n$ of function-germs at the point $0 \in \mathbb{C}^n$.

Definition. Two function-germs at zero are said to be *equivalent* if one is taken into the other by a biholomorphic change of coordinates that keeps the point zero fixed.

This notion of equivalence can be alternatively described as follows. Let $\mathcal{D}_n$ denote the group of germs of biholomorphic maps $g: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. This group acts on the space $\mathcal{O}_n$ of function-germs by the rule $g(f) = f \circ g^{-1}$, where $f \in \mathcal{O}_n$, $g \in \mathcal{D}_n$. The orbits of this action are exactly the equivalence classes of function-germs.

Definition. Two critical points are said to be *equivalent* if the function-germs that define them are equivalent. The equivalence class of a function-germ at a critical point is called a *singularity*.

Example. The functions $f_1 = x^2$ and $f_2 = cx^2$, with $c \neq 0$ a constant, have the same singularity at the point $x = 0$, since $f_1 = f_2 \circ g^{-1}$, where g is the diffeomorphism given by $g(x) = \sqrt{c}x$.

Clearly, equivalent critical points have equal coranks; hence the corank is the simplest invariant of a singularity.

The behavior of a function in the neighborhood of a nondegenerate critical point is described by the Morse lemma.

Theorem ([246]). *In a neighborhood of a nondegenerate critical point $a \in \mathbb{C}^n$ of the function f there exists a coordinate system in which f has the form*

$$f(x) = x_1^2 + \cdots + x_n^2 + f(a).$$

Remark. In the case of a smooth real-valued function $f: \mathbb{R}^n \to \mathbb{R}$ and real diffeomorphisms there is an additional invariant of a singularity, namely, the inertia index λ of the quadratic form defined by the second differential. In this case, in a neighborhood of a nondegenerate critical point a the function is reducible to the form

$$f(x) = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2 + f(a).$$

1.3. Stable Equivalence. For degenerate critical points, a generalization of the preceding result, the parametric Morse lemma, holds true.

Theorem ([12]). *In a neighborhood of the critical point* 0 *of corank k a holomorphic function $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is equivalent to a function of the form*

$$\varphi(x_1, \ldots, x_k) + x_{k+1}^2 + \cdots + x_n^2,$$

where the second differential of φ at zero is equal to zero: $\varphi \in \mathfrak{m}^3 \subset \mathcal{O}_n$.

[Here and in what follows $\mathfrak{m}$ denotes the ideal of the function-germs vanishing at the origin.]

This permits one to define an equivalence relation for critical points of (functions of) different numbers of variables.

Definition. Two function-germs $f: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g: (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ are said to be *stably equivalent* if they become equivalent after the addition of nondegenerate quadratic forms in supplementary variables:

$$f(x_1, \ldots, x_n) + x_{n+1}^2 + \cdots + x_k^2 \sim g(y_1, \ldots, y_m) + y_{m+1}^2 + \cdots + y_k^2.$$

Theorem ([374]). *Two functions of the same number of variables are stably equivalent if and only if they are equivalent.*

Thus, the passage to stable equivalence does not affect the classification of critical points of functions of a fixed number of variables and allows one to compare degeneracies of critical points of functions of different numbers of variables.

Example. The function-germs $f(x) = x^3$ and $g(x, y, z) = x^3 + yz$ are stably equivalent at zero.