# An Introduction to Aircraft Structural Analysis

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## PART B ANALYSIS OF AIRCRAFT STRUCTURES

# Preface

During my experience of teaching aircraft structures, I have felt the need for a textbook written specifically for students of aeronautical engineering. Although there have been a number of excellent books written on the subject, they are now either out of date or too specialized in content to fulfill the requirements of an undergraduate textbook. With that in mind, I wrote *Aircraft Structures for Engineering Students*, the text on which this one is based. Users of that text have supplied many useful comments to the publisher, including comments that a briefer version of the book might be desirable, particularly for programs that do not have the time to cover all the material in the "big" book. That feedback, along with a survey done by the publisher, resulted in this book, *An Introduction to Aircraft Structural Analysis*, designed to meet the needs of more time-constrained courses.

Much of the content of this book is similar to that of *Aircraft Structures for Engineering Students*, but the chapter on "Vibration of Structures" has been removed since this is most often covered in a separate standalone course. The topic of Aeroelasticity has also been removed, leaving detailed treatment to the graduate-level curriculum. The section on "Structural Loading and Discontinuities" remains in the big book but not this "intro" one. While these topics help develop a deeper understanding of load transfer and constraint effects in aircraft structures, they are often outside the scope of an undergraduate text. The reader interested in learning more on those topics should refer to the "big" book. In the interest of saving space, the appendix on "Design of a Rear Fuselage" is available for download from the book's companion Web site. Please visit *www.elsevierdirect.com* and search on "Megson" to find the Web site and the downloadable content.

Supplementary materials, including solutions to end-of-chapter problems, are available for registered instructors who adopt this book as a course text. Please visit *www.textbooks.elsevier.com* for information and to register for access to these resources.

The help of Tom Lacy, Associate Professor of Mechanical and Aerospace Engineering at Mississippi State University, is gratefully acknowledged in the development of this book.

T.H.G. Megson

**Supporting material accompanying this book**

A full set of worked solutions for this book are available for teaching purposes.

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PART

# A

# Fundamentals of Structural Analysis

CHAPTER

# Basic Elasticity

1

We shall consider, in this chapter, the basic ideas and relationships of the theory of elasticity. The treatment is divided into three broad sections: stress, strain, and stress–strain relationships. The third section is deferred until the end of the chapter to emphasize that the analysis of stress and strain—for example, the equations of equilibrium and compatibility—does not assume a particular stress–strain law. In other words, the relationships derived in Sections 1.1 through 1.14 inclusive are applicable to nonlinear as well as linear elastic bodies.

## 1.1 STRESS

Consider the arbitrarily shaped, three-dimensional body shown in Fig. 1.1. The body is in equilibrium under the action of externally applied forces $P_1, P_2, \ldots$, and is assumed to comprise a continuous and deformable material so that the forces are transmitted throughout its volume. It follows that at any internal point O, there is a resultant force $\delta P$. The particle of material at O subjected to the force $\delta P$ is in equilibrium so that there must be an equal but opposite force $\delta P$ (shown dotted in Fig. 1.1) acting on the particle at the same time. If we now divide the body by any plane *nn* containing O, then these two

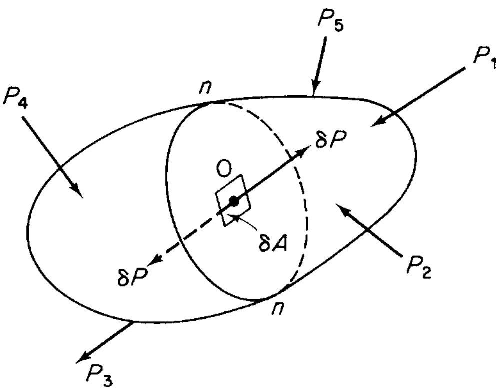


**Fig. 1.1**

Internal force at a point in an arbitrarily shaped body.


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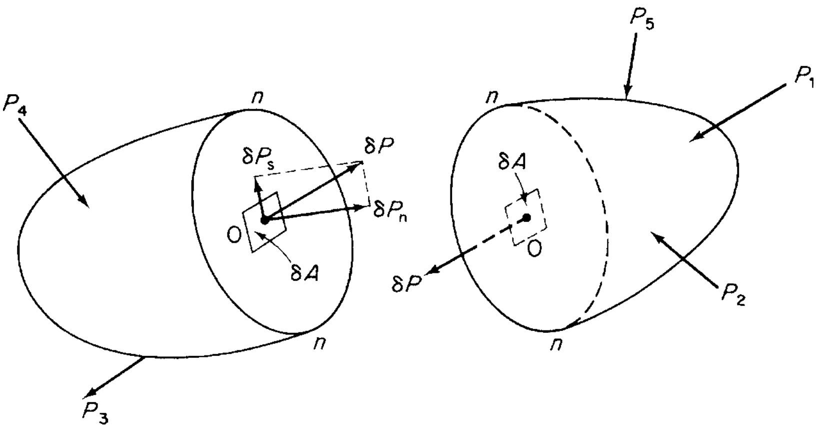


**Fig. 1.2**

Internal force components at the point O.

forces $\delta P$ may be considered uniformly distributed over a small area $\delta A$ of each face of the plane at the corresponding point O, as in Fig. 1.2. The *stress* at O is then defined by the equation

$$\text{Stress} = \lim_{\delta A \to 0} \frac{\delta P}{\delta A} \tag{1.1}$$

The directions of the forces $\delta P$ in Fig. 1.2 are such that they produce *tensile* stresses on the faces of the plane *nn*. It must be realized here that while the direction of $\delta P$ is absolute, the choice of plane is arbitrary so that although the direction of the stress at O will always be in the direction of $\delta P$, its magnitude depends on the actual plane chosen, since a different plane will have a different inclination and therefore a different value for the area $\delta A$. This may be more easily understood by reference to the bar in simple tension in Fig. 1.3. On the cross-sectional plane *mm*, the uniform stress is given by $P/A$, while on the inclined plane $m'm'$, the stress is of magnitude $P/A'$. In both cases, the stresses are parallel to the direction of $P$.

Generally, the direction of $\delta P$ is not normal to the area $\delta A$, in which case it is usual to resolve $\delta P$ into two components: one, $\delta P_n$, normal to the plane and the other, $\delta P_s$, acting in the plane itself (see Fig. 1.2). Note that in Fig. 1.2 the plane containing $\delta P$ is perpendicular to $\delta A$. The stresses associated with these components are a *normal* or *direct stress* defined as

$$\sigma = \lim_{\delta A \to 0} \frac{\delta P_n}{\delta A} \tag{1.2}$$

and a *shear stress* defined as

$$\tau = \lim_{\delta A \to 0} \frac{\delta P_s}{\delta A} \tag{1.3}$$

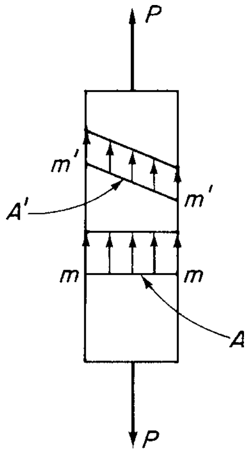


**Fig. 1.3**

Values of stress on different planes in a uniform bar.

The resultant stress is computed from its components by the normal rules of vector addition, namely

$$\text{Resultant stress} = \sqrt{\sigma^2 + \tau^2}$$

Generally, however, as just indicated, we are interested in the separate effects of $\sigma$ and $\tau$.

However, to be strictly accurate, stress is not a vector quantity, for, in addition to magnitude and direction, we must specify the plane on which the stress acts. Stress is therefore a *tensor*, with its complete description depending on the two vectors of force and surface of action.

## 1.2 NOTATION FOR FORCES AND STRESSES

It is usually convenient to refer the state of stress at a point in a body to an orthogonal set of axes O$xyz$. In this case, we cut the body by planes parallel to the direction of the axes. The resultant force $\delta P$ acting at the point O on one of these planes may then be resolved into a normal component and two in-plane components as shown in Fig. 1.4, thereby producing one component of direct stress and two components of shear stress.

The direct stress component is specified by reference to the plane on which it acts, but the stress components require a specification of direction in addition to the plane. We therefore allocate a single subscript to direct stress to denote the plane on which it acts and two subscripts to shear stress, the first specifying the plane and the second direction. Therefore, in Fig. 1.4, the shear stress components are $\tau_{zx}$ and $\tau_{zy}$ acting on the $z$ plane and in the $x$ and $y$ directions, respectively, while the direct stress component is $\sigma_z$.

We may now completely describe the state of stress at a point O in a body by specifying components of shear and direct stresses on the faces of an element of side $\delta x$, $\delta y$, and $\delta z$, formed at O by the cutting planes as indicated in Fig. 1.5.

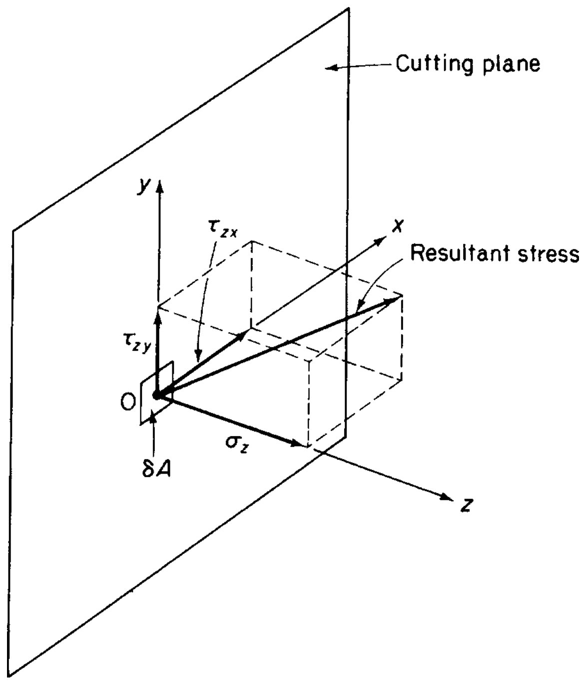


**Fig. 1.4**

Components of stress at a point in a body.

The sides of the element are infinitesimally small so that the stresses may be assumed to be uniformly distributed over the surface of each face. On each of the opposite faces, there will be, to a first simplification, equal but opposite stresses.

We shall now define the directions of the stresses in Fig. 1.5 as positive so that normal stresses directed away from their related surfaces are tensile and positive, and opposite compressive stresses are negative. Shear stresses are positive when they act in the positive direction of the relevant axis in a plane on which the direct tensile stress is in the positive direction of the axis. If the tensile stress is in the opposite direction, then positive shear stresses are in directions opposite to the positive directions of the appropriate axes.

Two types of external forces may act on a body to produce the internal stress system we have already discussed. Of these, *surface forces* such as $P_1, P_2, \ldots$, or hydrostatic pressure are distributed over the surface area of the body. The surface force per unit area may be resolved into components parallel to our orthogonal system of axes, and these are generally given the symbols $\overline{X}$, $\overline{Y}$, and $\overline{Z}$. The second force system derives from gravitational and inertia effects, and the forces are known as *body forces*. These are distributed over the volume of the body, and the components of body force per unit volume are designated $X$, $Y$, and $Z$.

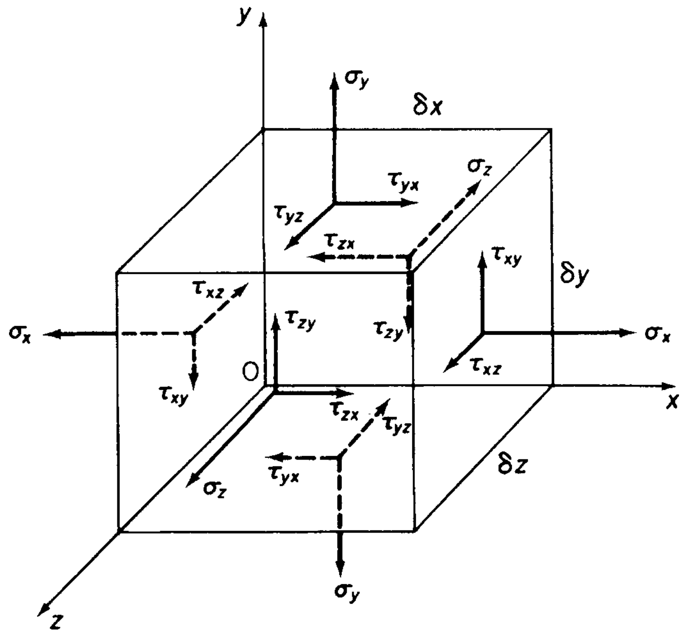


**Fig. 1.5**

Sign conventions and notation for stresses at a point in a body.

## 1.3 EQUATIONS OF EQUILIBRIUM

Generally, except in cases of uniform stress, the direct and shear stresses on opposite faces of an element are not equal as indicated in Fig. 1.5 but differ by small amounts. Therefore if, say, the direct stress acting on the $z$ plane is $\sigma_z$, then the direct stress acting on the $z+\delta z$ plane is, from the first two terms of a Taylor's series expansion, $\sigma_z + (\partial\sigma_z/\partial z)\delta z$. We now investigate the equilibrium of an element at some internal point in an elastic body where the stress system is obtained by the method just described.

In Fig. 1.6, the element is in equilibrium under forces corresponding to the stresses shown and the components of body forces (not shown). Surface forces acting on the boundary of the body, although contributing to the production of the internal stress system, do not directly feature in the equilibrium equations.

Taking moments about an axis through the center of the element parallel to the $z$ axis

$$\tau_{xy}\delta y\delta z\frac{\delta x}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\delta x\right)\delta y\delta z\frac{\delta x}{2} - \tau_{yx}\delta x\delta z\frac{\delta y}{2} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\delta y\right)\delta x\delta z\frac{\delta y}{2} = 0$$

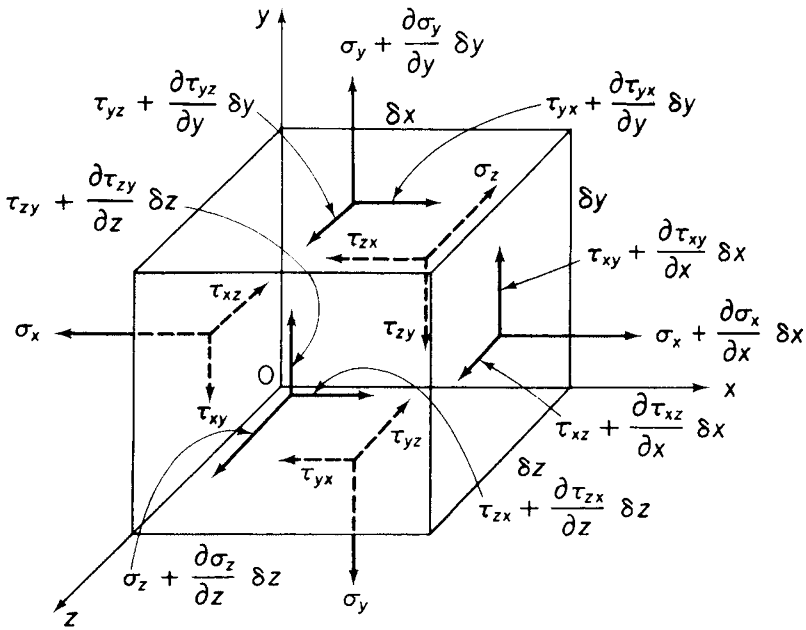


**Fig. 1.6**

Stresses on the faces of an element at a point in an elastic body.

which simplifies to

$$\tau_{xy}\delta y\delta z\delta x + \frac{\partial \tau_{xy}}{\partial x}\delta y\delta z\frac{(\delta x)^2}{2} - \tau_{yx}\delta x\delta z\delta y - \frac{\partial \tau_{yx}}{\partial y}\delta x\,\delta z\frac{(\delta y)^2}{2} = 0$$

Dividing by $\delta x\delta y\delta z$ and taking the limit as $\delta x$ and $\delta y$ approach zero.

Similarly,

$$\left.\begin{aligned}\tau_{xy} &= \tau_{yx}\\ \tau_{xz} &= \tau_{zx}\\ \tau_{yz} &= \tau_{zy}\end{aligned}\right\} \tag{1.4}$$

We see, therefore, that a shear stress acting on a given plane ($\tau_{xy}, \tau_{xz}, \tau_{yz}$) is always accompanied by an equal *complementary shear stress* ($\tau_{yx}, \tau_{zx}, \tau_{zy}$) acting on a plane perpendicular to the given plane and in the opposite sense.

Now considering the equilibrium of the element in the $x$ direction

$$\begin{aligned}\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\delta x\right)\delta y\,\delta z - \sigma_x\delta y\delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\delta y\right)\delta x\delta z\\ - \tau_{yx}\delta x\delta z + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\delta z\right)\delta x\delta y\\ - \tau_{zx}\delta x\delta y + X\delta x\delta y\delta z = 0\end{aligned}$$

which gives

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

Or, writing $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$ from Eq. (1.4).

Similarly,

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0 \end{aligned}\right\} \tag{1.5}$$

The *equations of equilibrium* must be satisfied at all interior points in a deformable body under a three-dimensional force system.

## 1.4 PLANE STRESS

Most aircraft structural components are fabricated from thin metal sheet so that stresses across the thickness of the sheet are usually negligible. Assuming, say, that the $z$ axis is in the direction of the thickness, then the three-dimensional case of Section 1.3 reduces to a two-dimensional case in which $\sigma_z, \tau_{xz}$, and $\tau_{yz}$ are all zero. This condition is known as *plane stress*; the equilibrium equations then simplify to

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + Y = 0 \end{aligned}\right\} \tag{1.6}$$

## 1.5 BOUNDARY CONDITIONS

The equations of equilibrium (1.5) (and also (1.6) for a two-dimensional system) satisfy the requirements of equilibrium at all internal points of the body. Equilibrium must also be satisfied at all positions on the boundary of the body where the components of the surface force per unit area are $\overline{X}$, $\overline{Y}$, and $\overline{Z}$. The triangular element of Fig. 1.7 at the boundary of a two-dimensional body of unit thickness is then in equilibrium under the action of surface forces on the elemental length AB of the boundary and internal forces on internal faces AC and CB.

Summation of forces in the $x$ direction gives

$$\overline{X}\delta s - \sigma_x \delta y - \tau_{yx}\delta x + X\frac{1}{2}\delta x \delta y = 0$$

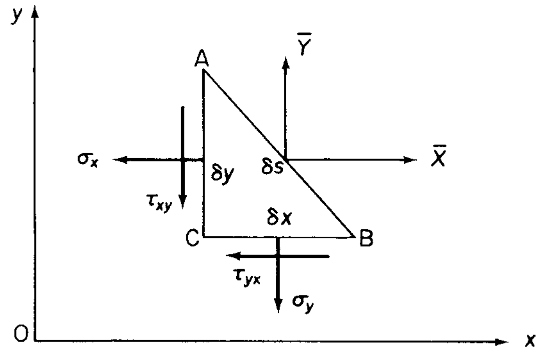


**Fig. 1.7**

Stresses on the faces of an element at the boundary of a two-dimensional body.

which, by taking the limit as $\delta x$ approaches zero, becomes

$$\bar{X} = \sigma_x \frac{dy}{ds} + \tau_{yx} \frac{dx}{ds}$$

The derivatives $dy/ds$ and $dx/ds$ are the direction cosines $l$ and $m$ of the angles that a normal to AB makes with the $x$ and $y$ axes, respectively. It follows that

$$\bar{X} = \sigma_x l + \tau_{yx} m$$

and in a similar manner,

$$\bar{Y} = \sigma_y m + \tau_{xy} l$$

A relatively simple extension of this analysis produces the boundary conditions for a three-dimensional body, namely

$$\left.\begin{aligned} \bar{X} &= \sigma_x l + \tau_{yx} m + \tau_{zx} n \\ \bar{Y} &= \sigma_y m + \tau_{xy} l + \tau_{zy} n \\ \bar{Z} &= \sigma_z n + \tau_{yz} m + \tau_{xz} l \end{aligned}\right\} \tag{1.7}$$

where $l$, $m$, and $n$ become the direction cosines of the angles that a normal to the surface of the body makes with the $x$,$y$, and $z$ axes, respectively.

## 1.6 DETERMINATION OF STRESSES ON INCLINED PLANES

The complex stress system of Fig. 1.6 is derived from a consideration of the actual loads applied to a body and is referred to a predetermined, though arbitrary, system of axes. The values of these stresses may not give a true picture of the severity of stress at that point, so it is necessary to investigate the

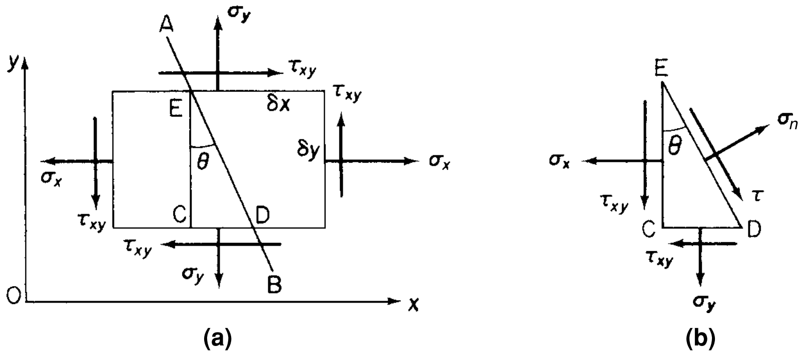


**Fig. 1.8**

(a) Stresses on a two-dimensional element; (b) stresses on an inclined plane at the point.

state of stress on other planes on which the direct and shear stresses may be greater. We shall restrict the analysis to the two-dimensional system of plane stress defined in Section 1.4.

Figure 1.8(a) shows a complex stress system at a point in a body referred to axes O$x$, O$y$. All stresses are positive as defined in Section 1.2. The shear stresses $\tau_{xy}$ and $\tau_{yx}$ were shown to be equal in Section 1.3. We now, therefore, designate them both $\tau_{xy}$. The element of side $\delta x, \delta y$ and of unit thickness is small, so stress distributions over the sides of the element may be assumed to be uniform. Body forces are ignored, since their contribution is a second-order term.

Suppose that we want to find the state of stress on a plane AB inclined at an angle $\theta$ to the vertical. The triangular element EDC formed by the plane and the vertical through E is in equilibrium under the action of the forces corresponding to the stresses shown in Fig. 1.8(b), where $\sigma_n$ and $\tau$ are the direct and shear components of the resultant stress on AB. Then, resolving forces in a direction perpendicular to ED, we have

$$\sigma_n \mathrm{ED} = \sigma_x \mathrm{EC} \cos\theta + \sigma_y \mathrm{CD} \sin\theta + \tau_{xy} \mathrm{EC} \sin\theta + \tau_{xy} \mathrm{CD} \cos\theta$$

Dividing by ED and simplifying

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta \tag{1.8}$$

Now resolving forces parallel to ED,

$$\tau \mathrm{ED} = \sigma_x \mathrm{EC} \sin\theta - \sigma_y \mathrm{CD} \cos\theta - \tau_{xy} \mathrm{EC} \cos\theta + \tau_{xy} \mathrm{CD} \sin\theta$$

Again dividing by ED and simplifying,

$$\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \tag{1.9}$$

**Example 1.1**

A cylindrical pressure vessel has an internal diameter of 2 m and is fabricated from plates 20 mm thick. If the pressure inside the vessel is 1.5 N/mm$^2$ and, in addition, the vessel is subjected to an axial tensile load of 2500 kN, calculate the direct and shear stresses on a plane inclined at an angle of 60° to the axis of the vessel. Calculate also the maximum shear stress.

The expressions for the longitudinal and circumferential stresses produced by the internal pressure may be found in any text on stress analysis and are

$$\text{Longitudinal stress } (\sigma_x) = \frac{pd}{4t} = 1.5 \times 2 \times 10^3/4 \times 20 = 37.5\,\text{N/mm}^2$$
$$\text{Circumferential stress } (\sigma_y) = \frac{pd}{2t} = 1.5 \times 2 \times 10^3/2 \times 20 = 75\,\text{N/mm}^2$$

The direct stress due to the axial load contributes to $\sigma_x$ and is given by

$$\sigma_x \text{ (axial load)} = 2500 \times 10^3/\pi \times 2 \times 10^3 \times 20 = 19.9\,\text{N/mm}^2$$

A rectangular element in the wall of the pressure vessel is then subjected to the stress system shown in Fig. 1.9. Note that there are no shear stresses acting on the $x$ and $y$ planes; in this case, $\sigma_x$ and $\sigma_y$ then form a *biaxial* stress system.

The direct stress, $\sigma_n$, and shear stress, $\tau$, on the plane AB that makes an angle of 60° with the axis of the vessel may be found from first principles by considering the equilibrium of the triangular element ABC or by direct substitution in Eqs. (1.8) and (1.9). Note that in the latter case, $\theta = 30°$ and $\tau_{xy} = 0$. Then,

$$\sigma_n = 57.4\cos^2 30° + 75\sin^2 30° = 61.8\,\text{N/mm}^2$$
$$\tau = (57.4 - 75)(\sin(2 \times 30°))/2 = -7.6\,\text{N/mm}^2$$

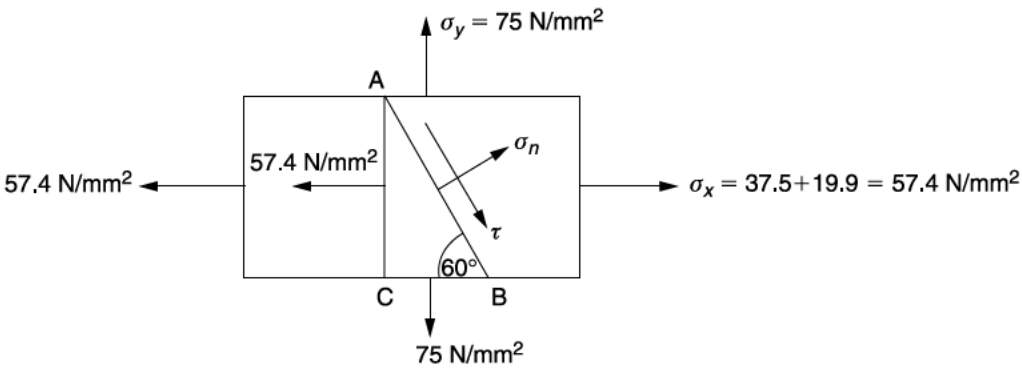


**Fig. 1.9**

Element of Example 1.1.

The negative sign for $\tau$ indicates that the shear stress is in the direction BA and not in AB.

From Eq. (1.9) when $\tau_{xy} = 0$,

$$\tau = (\sigma_x - \sigma_y)(\sin 2\theta)/2 \tag{i}$$

The maximum value of $\tau$ therefore occurs when $\sin 2\theta$ is a maximum—that is, when $\sin 2\theta = 1$ and $\theta = 45°$. Then, substituting the values of $\sigma_x$ and $\sigma_y$ in Eq. (i),

$$\tau_{max} = (57.4 - 75)/2 = -8.8\,\text{N/mm}^2$$

**Example 1.2**

A cantilever beam of solid, circular cross section supports a compressive load of 50 kN applied to its free end at a point 1.5 mm below a horizontal diameter in the vertical plane of symmetry together with a torque of 1200 Nm (Fig. 1.10). Calculate the direct and shear stresses on a plane inclined at 60° to the axis of the cantilever at a point on the lower edge of the vertical plane of symmetry.

The direct loading system is equivalent to an axial load of 50 kN together with a bending moment of $50 \times 10^3 \times 1.5 = 75\,000$ N/mm in a vertical plane. Therefore, at any point on the lower edge of the vertical plane of symmetry, there are compressive stresses due to the axial load and bending moment which act on planes perpendicular to the axis of the beam and are given, respectively, by Eqs. (1.2) and (15.9):

$$\sigma_x \text{ (axial load)} = 50 \times 10^3/\pi \times (60^2/4) = 17.7\,\text{N/mm}^2$$

$$\sigma_x \text{ (bending moment)} = 75\,000 \times 30/\pi \times (60^4/64) = 3.5\,\text{N/mm}^2$$

The shear stress, $\tau_{xy}$, at the same point due to the torque is obtained from Eq. (iv) in Example 3.1, that is,

$$\tau_{xy} = 1200 \times 10^3 \times 30/\pi \times (60^4/32) = 28.3\,\text{N/mm}^2$$

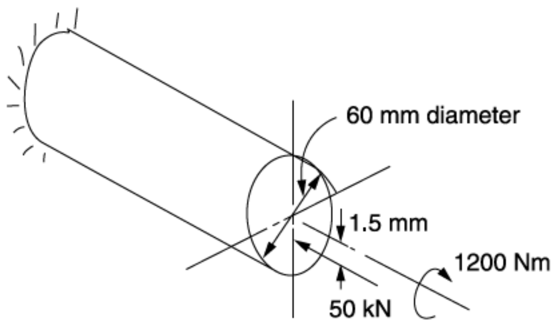


**Fig. 1.10**

Cantilever beam of Example 1.2.