Aerodynamics of the Airplane

Hermann Schlichting and Erich Truckenbrodt

Translated by Heinrich J. Ramm
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Only a very few comprehensive presentations of the scientific fundamentals of the aerodynamics of the airplane have ever been published. The present book is an English translation of the two-volume work "Aerodynamik des Flugzeuges," which has already appeared in a second edition in the original German. In this book we treat exclusively the aerodynamic forces that act on airplane components—and thus on the whole airplane—during its motion through the earth's atmosphere (aerodynamics of the airframe). These aerodynamic forces depend in a quite complex manner on the geometry, speed, and motion of the airplane and on the properties of air. The determination of these relationships is the object of the study of the aerodynamics of the airplane. Moreover, these relationships provide the absolutely necessary basis for determining the flight mechanics and many questions of the structural requirements of the airplane, and thus for airplane design. The aerodynamic problems related to airplane propulsion (power plant aerodynamics) and the theory of the modes of motion of the airplane (flight mechanics) are not treated in this book.

The study of the aerodynamics of the airplane requires a thorough knowledge of aerodynamic theory. Therefore, it was necessary to include in the German edition a rather comprehensive outline of fluid mechanic theory. In the English edition this section has been eliminated because there exist a sufficient number of pertinent works in English on the fundamentals of fluid mechanic theory.

Chapter 1 serves as an introduction. It describes the physical properties of air and of the atmosphere, and outlines the basic aerodynamic behavior of the airplane. The main portion of the book consists of three major divisions. In the first division (Part 1), Chaps. 2-4 cover the aerodynamics of the airfoil. In the second division (Part 2), Chaps. 5 and 6 consider the aerodynamics of the fuselage and of the wing-fuselage system. Finally, in the third division (Part 3), Chaps. 7 and 8 are devoted to the problems of the aerodynamics of the stability and control systems (empennage, flaps, and control surfaces). In Parts 2 and 3, the interactions among the individual parts of the airplane, that is, the aerodynamic interference, are given special attention.

Specifically, the following brief outline describes the chapters that deal with the intrinsic problems of the aerodynamics of the airplane: Part 1 contains, in Chap. 2, the profile theory of incompressible flow, including the influence of friction on the profile
Chapter 3 gives a comprehensive account of three-dimensional wing theory for incompressible flow (lifting-line and lifting-surface theory). In addition to linear airfoil theory, nonlinear wing theory is treated because it is of particular importance for modern airplanes (slender wings). The theory for incompressible flow is important not only in the range of moderate flight velocities, at which the compressibility of the air may be disregarded, but even at higher velocities, up to the speed of sound—that is, at all Mach numbers lower than unity—the pressure distribution of the wings can be related to that for incompressible flow by means of the Prandtl-Glauert transformation. In Chap. 4, the wing in compressible flow is treated. Here, in addition to profile theory, the theory of the wing of finite span is discussed at some length. The chapter is subdivided into the aerodynamics of the wing at subsonic and supersonic, and at transonic and hypersonic incident flow. The latter two cases are treated only briefly. Results of systematic experimental studies on simple wing forms in the subsonic, transonic, and supersonic ranges are given for the qualification of the theoretical results. Part 2 begins in Chap. 5 with the aerodynamics of the fuselage without interference at subsonic and supersonic speeds. In Chap. 6, a rather comprehensive account is given of the quite complex, but for practical cases very important, aerodynamic interference of wing and fuselage (wing-fuselage system). It should be noted that the difficult and complex theory of supersonic flow could be treated only superficially. In this chapter, a special section is devoted to slender flight articles. Here, some recent experimental results, particularly for slender wing-fuselage systems, are reported. In Part 3, Chaps. 7 and 8, the aerodynamic questions of importance to airplane stability and control are treated. Here, the aerodynamic interferences of wing and wing-fuselage systems are of decisive significance. Experimental results on the maximum lift and the effect of landing flaps (air brakes) are given. The discussions of this part of the aerodynamics of the airplane refer again to subsonic and supersonic incident flow.

A comprehensive list of references complements each chapter. These lists, as well as the bibliography at the end of the book, have been updated from the German edition to include the most recent publications.

Although the book is addressed primarily to students of aeronautics, it has been written as well with the engineers and scientists in mind who work in the aircraft industry and who do research in this field. We have endeavored to emphasize the theoretical approach to the problems, but we have tried to do this in a manner easily understandable to the engineer. Actually, through proper application of the laws of modern aerodynamics it is possible today to derive a major portion of the aerodynamics of the airplane from purely theoretical considerations. The very comprehensive experimental material, available in the literature, has been included only as far as necessary to create a better physical concept and to check the theory. We wanted to emphasize that decisive progress has been made not through accumulation of large numbers of experimental results, but rather through synthesis of theoretical considerations with a few basic experimental results. Through numerous detailed examples, we have endeavored to enhance the reader’s comprehension of the theory.

Hermann Schlichting
Erich Truckenbrodt
NOMENCLATURE

MATERIAL CONSTANTS

\( \rho \)  
density of air (mass of unit volume)

\( g \)  
gravitational acceleration

\( c_p, c_v \)  
specific heats at constant pressure and constant volume, respectively

\( \gamma = \frac{c_p}{c_v} \)  
isentropic exponent

\( a = \sqrt{\gamma \rho / \dot{\rho}} \)  
speed of sound

\( \mu \)  
coefficient of dynamic viscosity

\( v = \frac{\mu}{\rho} \)  
coefficient of kinematic viscosity

\( R \)  
gas constant

\( T \)  
absolute temperature (K)

\( t \)  
temperature (°C)

FLOW QUANTITIES

\( p \)  
pressure (normal force per unit area)

\( \tau \)  
shear stress (tangential force per unit area)

\( u, v, w \)  
velocity components in Cartesian (rectangular) coordinates

\( u_r, w_r, w_\theta \)  
velocity components in cylindrical coordinates

\( V, U_\infty \)  
velocity of incident flow

\( W_C \)  
velocity on profile contour

\( w_i \)  
induced downwash velocity, positive in the direction of the negative z axis
X NOMENCLATURE

\[ q = (q/2)V^2 \]  
\[ q_\infty = (q_\infty/2)U_\infty^2 \]  
\[ Re = Vl/\nu \]  
\[ Ma = V/a \]  
\[ Ma_\infty = U_\infty/a_\infty \]  
\[ \mu \]  
\[ \delta_1 \]  
\[ \Gamma \]  
\[ \gamma = \Gamma/bU_\infty \]  
\[ k, k_x, k_y \]  
\[ q \]  
\[ m \]  
\[ \Phi \]

dynamic (impact) pressure  
dynamic (impact) pressure of undisturbed flow  
Reynolds number  
Mach number  
Mach number of undisturbed flow  
drag-critical Mach number  
Mach angle  
displacement thickness of boundary layer  
circulation  
dimensionless circulation  
vortex density  
source strength  
dipole strength  
velocity potential

GEOMETRIC QUANTITIES

\[ x, y, z \]

Cartesian (rectangular) coordinates: \( x = \) longitudinal axis, \( y = \) lateral axis, \( z = \) vertical axis

\[ \xi = x/s, \eta = y/s, \]  
\[ \zeta = z/s \]  

dimensionless rectangular coordinates  
trigonometric coordinate; \( \cos \theta = \eta \)

coordinates of wing leading (front) and trailing (rear) edges, \( x_0, x_{100} \), respectively

coordinates of quarter-point and three-quarter-point lines, \( x_{25}, x_{75} \), respectively

\[ A \]  
\[ A_F \]  
\[ A_H \]  
\[ A_V \]  
\[ b = 2s \]  
\[ b_F \]  
\[ b_H \]  
\[ A = b^2/A \]  
\[ A_{H}, A_{V} \]  

wing area  
fuselage cross-sectional area  
area of horizontal tail (surface)  
area of vertical tail (surface)  
wing span  
fuselage width  
span of horizontal tail (surface)  
aspect ratio of wing  
aspect ratios of horizontal and vertical tails (surface), respectively

\[ c \]  
\[ c_r, c_t \]  
\[ c_\mu = (2/A) \int_0^\infty c^2(y) \, dy \]  
\[ \lambda = c_t/c_r \]  
\[ l_F \]  
\[ c_f \]  
\[ \lambda_f = c_f/c \]  
\[ \eta_f \]

wing chord  
chord at wing root and wing tip, respectively  
wing reference chord  
wing taper  
fuselage length  
flap (control-surface) chord  
flap (control-surface) chord ratio  
flap deflection
NOMENCLATURE

\[ \varphi \]
- sweepback angle of wing

\[ \gamma \]
- leading edge semiangle of delta wing (Fig. 4-59)

\[ m = \tan \gamma / \tan \mu \]
- parameter (Fig. 4-59); \( m < 1 \): subsonic flow edge, \( m > 1 \): supersonic flow edge

\[ \varepsilon \]
- twist angle

\[ \nu \]
- angle of wing dihedral

\[ N_{25} \]
- geometric neutral point

\[ t \]
- profile thickness

\[ \delta = t/c \]
- thickness ratio of wing

\[ h \]
- profile camber

\[ \xi_t \]
- (maximum) thickness position

\[ \xi_h \]
- (maximum) camber (height) position

\[ z(s) \]
- skeleton (mean camber) line coordinate

\[ z(r) \]
- teardrop profile coordinate

\[ d_{F_{\text{max}}} \]
- maximum fuselage diameter

\[ \delta_F = d_{F_{\text{max}}}/l_F \]
- fuselage thickness ratio

\[ \eta_F = b_F/b \]
- relative fuselage width

\[ D = 2R \]
- diameter of axisymmetric fuselage

\[ z_0 \]
- wing vertical position

\[ r_H \]
- lever arm of horizontal tail (= distance between geometric neutral points of the wing and the horizontal tail)

\[ \varepsilon_H \]
- setting angle of horizontal stabilizer (tail)

\[ r_V \]
- lever arm of vertical tail (= distance between geometric neutral points of the wing and the vertical tail)

AERODYNAMIC QUANTITIES (see Fig. 1-6)

\[ \alpha \]
- angle of attack (incidence)

\[ \beta \]
- angle of sideslip (yaw)

\[ \omega_x, \omega_y, \omega_z \]
- components of angular velocities in rectangular coordinates during rotary motion of the airplane

\[ \Omega_x = \omega_x s/V, \]
\[ \Omega_y = \omega_y c \mu / V, \]
\[ \Omega_z = \omega_z s/V \]
- components of the dimensionless angular velocities

\[ L \]
- lift

\[ D \]
- drag

\[ Y \]
- side force

\[ M_x \]
- rolling moment

\[ M, M_y \]
- pitching moment

\[ M_z \]
- yawing moment

\[ D_I \]
- induced drag

\[ c_L \]
- lift coefficient

\[ c_D \]
- drag coefficient

\[ c_{Mx} \]
- rolling-moment coefficient
**NOMENCLATURE**

\[ c_M, c_{My} \]  
- pitching-moment coefficient

\[ c_{Mz} \]  
- yawing-moment coefficient

\[ c_l \]  
- local lift coefficient

\[ c_m \]  
- local pitching-moment coefficient

\[ c_{mf} \]  
- control-surface (hinge) moment coefficient

\[ c_{if} \]  
- flap (control-surface) load coefficient

\[ c_{Di} \]  
- coefficient of induced drag

\[ c_{Dp} \]  
- coefficient of profile drag

\[ c'_{L\infty} = (dc_L/d\alpha)_{\infty} \]  
- lift slope of wing of infinite span

\[ c_p = (p - p_{\infty})/q_{\infty} \]  
- pressure coefficient

\[ c_{ppl} \]  
- pressure coefficient of plane (two-dimensional) flow

\[ c_{pat} \]  
- critical pressure coefficient

\[ \Delta c_p = (p_I - p_u)q_{\infty} \]  
- coefficient of load distribution

\[ f = 2b/c'_{L\infty} \]  
- planform function

\[ k = \pi\Lambda/c'_{L\infty} \]  
- coefficient of elliptic wing

\[ \alpha_e \]  
- effective angle of attack

\[ \alpha_g \]  
- geometric angle of attack

\[ \alpha_i = wi/U_{\infty} \]  
- induced angle of attack

\[ \alpha_0 \]  
- zero-lift angle of attack

\[ \alpha_H = \alpha + \varepsilon_H + \alpha_w \]  
- angle of attack of the horizontal tail

\[ \alpha_w = w/U_{\infty} \]  
- downwash angle at the horizontal tail location

\[ N \]  
- aerodynamic neutral point

\[ x_N \]  
- position of aerodynamic neutral point

\[ \Delta x_N \]  
- distance between aerodynamic and geometric neutral points

\[ \beta_V \]  
- angle of flow incident on the vertical tail

\[ \beta_u = u/U_{\infty} \]  
- angle of sidewash at the station of the vertical tail

**DIMENSIONLESS STABILITY COEFFICIENTS**

**Coefficients of Yawed Flight**

\[ \partial c_Y/\partial \beta \]  
- side force due to sideslip

\[ \partial c_{Mx}/\partial \beta \]  
- rolling moment due to sideslip

\[ \partial c_{Mz}/\partial \beta \]  
- yawing moment due to sideslip

**Coefficients due to Angular Velocity**

\[ \partial c_Y/\partial \Omega_z \]  
- side force due to yaw rate

\[ \partial c_{Mx}/\partial \Omega_x \]  
- rolling moment due to roll rate

\[ \partial c_{Mx}/\partial \Omega_z \]  
- rolling moment due to yaw rate

\[ \partial c_{Mz}/\partial \Omega_x \]  
- yawing moment due to yaw rate

\[ \partial c_{Mz}/\partial \Omega_z \]  
- yawing moment due to roll rate

\[ \partial c_L/\partial \Omega_y \]  
- lift due to pitch rate

\[ \partial c_M/\partial \Omega_y \]  
- pitching moment due to pitch rate
## INDICES

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>wing data</td>
</tr>
<tr>
<td>$F$</td>
<td>fuselage data</td>
</tr>
<tr>
<td>$(W + F)$</td>
<td>data of wing-fuselage system</td>
</tr>
<tr>
<td>$H$</td>
<td>data of horizontal stabilizer</td>
</tr>
<tr>
<td>$V$</td>
<td>data of vertical stabilizer</td>
</tr>
<tr>
<td>$f$</td>
<td>data of flaps (control surfaces)</td>
</tr>
</tbody>
</table>
1-1 PROBLEMS OF AIRPLANE AERODYNAMICS

An airplane moves in the earth’s atmosphere. The state of motion of an airplane is determined by its weight, by the thrust of the power plant, and by the aerodynamic forces (or loads) that act on the airplane parts during their motion. For every state of motion at uniform velocity, the resultant of weight and thrust forces must be in equilibrium with the resultant of the aerodynamic forces. For the particularly simple state of motion of horizontal flight, the forces acting on the airplane are shown in Fig. 1-1. In this case, the equilibrium condition is reduced to the requirement that, in the vertical direction, the weight must be equal to the lift ($W = L$) and, in the horizontal direction, the thrust must be equal to the drag ($Th = D$). Here, lift $L$ and drag $D$ are the components of the aerodynamic force $R_1$ normal and parallel, respectively, to the flight velocity vector $V$. For nonuniform motion of the aircraft, inertia forces are to be added to these forces.

In this book we shall deal exclusively with aerodynamic forces that act on the individual parts, and thus on the whole aircraft, during motion. The most important parts of the airplane that contribute to the aerodynamic forces are wing, fuselage, control and stabilizing surfaces (tail unit or empennage, ailerons, and canard surfaces), and power plant. The aerodynamic forces depend in a quite complicated manner on the geometry of these parts, the flight speed, and the physical properties of the air (e.g., density, viscosity). It is the object of the study of the aerodynamics of the airplane to furnish information about these interrelations. Here, the following two problem areas have to be considered:

1. Determination of aerodynamic forces for a given geometry of the aircraft (direct problem)
2. Determination of the geometry of the aircraft for desired flow patterns (indirect problem)
2 INTRODUCTION

The object of flight mechanics is the determination of aircraft motion for given aerodynamic forces, known weight of the aircraft, and given thrust. This includes questions of both flight performance and flight conditions, such as control and stability of the aircraft. Flight mechanics is not a part of the problem area of this book. Also, the field of aeroelasticity, that is, the interactions of aerodynamic forces with elastic forces during deformation of aircraft parts, will not be treated.

The science of the aerodynamic forces of airplanes, to be treated here, forms the foundation for both flight mechanics and many questions of aircraft design and construction.

1-2 PHYSICAL PROPERTIES OF AIR

1-2-1 Basic Facts

In fluid mechanics, some physical properties of the fluid are important, for example, density and viscosity. With regard to aircraft operation in the atmosphere, changes of these properties with altitude are of particular importance. These physical properties of the earth's atmosphere directly influence aircraft aerodynamics and consequently, indirectly, the flight mechanics. In the following discussions the fluid will be considered to be a continuum.

The density $\rho$ is defined as the mass of the unit volume. It depends on both pressure and temperature. Compressibility is a measure of the degree to which a fluid can be compressed under the influence of external pressure forces. The compressibility of gases is much greater than that of liquids. Compressibility
therefore has to be taken into account when changes in pressure resulting from a particular flow process lead to noticeable changes in density.

Viscosity is related to the friction forces within a streaming fluid, that is, to the tangential forces transmitted between ambient volume elements. The viscosity coefficient of fluids changes rather drastically with temperature.

In many technical applications, viscous forces can be neglected in order to simplify the laws of fluid dynamics (inviscid flow). This is done in the theory of lift of airfoils (potential flow). To determine the drag of bodies, however, the viscosity has to be considered (boundary-layer theory). The considerable increase in flight speed during the past decades has led to problems in aircraft aerodynamics that require inclusion of the compressibility of the air and often, simultaneously, the viscosity. This is the case when the flight speed becomes comparable to the speed of sound (gas dynamics). Furthermore, the dependence of the physical properties of air on the altitude must be known. Some quantitative data will now be given for density, compressibility, and viscosity of air.

1-2-2 Material Properties

**Density** The density of a gas (mass/volume), with the dimensions kg/m$^3$ or Ns$^2$/m$^4$, depends on pressure and temperature. The relationship between density $\rho$, pressure $p$, and absolute temperature $T$ is given by the thermal equation of state for ideal gases

$$ p = \rho RT \quad (1-1a) $$

$$ R = 287 \frac{J}{\text{kg} \cdot \text{K}} \text{ (air)} \quad (1-1b) $$

where $R$ is the gas constant. Of the various possible changes of state of a gas, of particular importance is the adiabatic-reversible (isentropic) change in which pressure and density are related by

$$ \frac{p}{\rho^\gamma} = \text{const} \quad (1-2) $$

Here $\gamma$ is the isentropic exponent, with

$$ \gamma = \frac{c_p}{c_v} \quad (1-3a) $$

$$ = 1.405 \text{ (air)} \quad (1-3b) $$

c$_p$ and c$_v$ are the specific heats at constant pressure and constant volume, respectively.

Very fast changes of state are adiabatic processes in very good approximation, because heat exchange with the ambient fluid elements is relatively slow and, therefore, of negligible influence on the process. In this sense, flow processes at high speeds can usually be considered to be fast changes of state. If such flows are steady, isentropic changes of state after Eq. (1-2) can be assumed. Unsteady-flow
processes (e.g., with shock waves) are not isentropic (anisentropic); they do not follow Eq. (1-2).

Across a normal compression shock, pressure and density are related by

\[ \frac{\rho_2}{\rho_1} = \frac{(\gamma - 1) + (\gamma + 1) \frac{p_2}{p_1}}{(\gamma + 1) + (\gamma - 1) \frac{p_2}{p_1}} \]

\[ \leq \frac{\gamma + 1}{\gamma - 1} = 5.93 \text{ (air)} \]

where the indices 1 and 2 indicate conditions before and behind the shock, respectively.

**Speed of sound** Since the pressure changes of acoustic vibrations in air are of such a high frequency that heat exchange with the ambient fluid elements is negligible, an isentropic change of state after Eq. (1-2) can be assumed for the compressibility law of air: \( p(\varrho) \). Then, with Laplace's formula, the speed of sound becomes

\[ a = \sqrt{\gamma \frac{p}{\varrho}} = \sqrt{\gamma RT} \]

\[ a_0 = 340 \text{ m/s (air)} \]

where for \( p/\varrho \) the value given by the equation of state for ideal gases, Eq. (1-1a), was taken. Note that the speed of sound is simply proportional to the square root of the absolute temperature. The value given in Eq. (1-5b) is valid for air of temperature \( t = 15^\circ \text{C} \) or \( T = 288 \text{ K} \).

**Viscosity** In flows of an inviscid fluid, no tangential forces (shear stresses) exist between ambient layers. Only normal forces (pressures) act on the flow. The theory of inviscid, incompressible flow has been developed mathematically in detail, giving, in many cases, a satisfactory description of the actual flow, for example, in computing airfoil lift at moderate flight velocities. On the other hand, this theory fails completely for the computation of body drag. This unacceptable result of the theory of inviscid flow is caused by the fact that both between the layers within the fluid and between the fluid and its solid boundary, tangential forces are transmitted that affect the flow in addition to the normal forces. These tangential or friction forces of a real fluid are the result of a fluid property, called the viscosity of the fluid. Viscosity is defined by Newton's elementary friction law of fluids as

\[ \tau = \mu \frac{du}{dy} \]

Here \( \tau \) is the shearing stress between adjacent layers, \( du/dy \) is the velocity gradient normal to the stream, and \( \mu \) is the dynamic viscosity of the fluid, having the dimensions \( \text{Ns/m}^2 \). It is a material constant that is almost independent of pressure but, in gases,
increases strongly with increasing temperature. In all flows governed by friction and inertia forces simultaneously, the quotient of viscosity $\mu$ and density $\rho$ plays an important role. It is called the kinematic viscosity $\nu$,

$$\nu = \frac{\mu}{\rho}$$  \hspace{1cm} (1-7)

and has the dimensions $m^2/s$. In Table 1-1 a few values for density $\rho$, dynamic viscosity $\mu$, and kinematic viscosity $\nu$ of air are given versus temperature at constant pressure.

1-2-3 Physical Properties of the Atmosphere

Changes of pressure, density, and viscosity of the air with altitude $z$ of the stationary atmosphere are important for aeronautics. These quantities depend on the vertical temperature distribution $T(z)$ in the atmosphere. At moderate altitude (up to about 10 km), the temperature decreases with increasing altitude, the temperature gradient $dT/dz$ varying between approximately $-0.5$ and $-1$ K per 100 m, depending on the weather conditions. At the higher altitudes, the temperature gradient varies strongly with altitude, with both positive and negative values occurring.

The data for the atmosphere given below are valid up to the boundary of the homosphere at an altitude of about 90 km. Here the gravitational acceleration is already markedly smaller than at sea level.

The pressure change for a step of vertical height $dz$ is, after the basic hydrostatic equation,

$$dp = -\rho g dz$$ \hspace{1cm} (1-8a)

$$= -\rho g_0 dH$$ \hspace{1cm} (1-8b)

where $H$ is called scale height.

Table 1-1 Density $\rho$, dynamic viscosity $\mu$, and kinematic viscosity $\nu$ of air versus temperature $t$ at constant pressure $p \approx 1$ atmosphere

<table>
<thead>
<tr>
<th>Temperature $t$ [°C]</th>
<th>Density $\rho$ [kg/m³]</th>
<th>Viscosity $\mu \cdot 10^6$ [kg/ms]</th>
<th>Kinematic viscosity $\nu \cdot 10^6$ [m²/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>1.39</td>
<td>15.6</td>
<td>11.3</td>
</tr>
<tr>
<td>-10</td>
<td>1.34</td>
<td>16.2</td>
<td>12.1</td>
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<td>0</td>
<td>1.29</td>
<td>16.8</td>
<td>13.0</td>
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<td>40</td>
<td>1.12</td>
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<td>17.0</td>
</tr>
<tr>
<td>60</td>
<td>1.06</td>
<td>20.3</td>
<td>19.2</td>
</tr>
<tr>
<td>80</td>
<td>0.99</td>
<td>21.5</td>
<td>21.7</td>
</tr>
<tr>
<td>100</td>
<td>0.94</td>
<td>22.9</td>
<td>24.5</td>
</tr>
</tbody>
</table>
The decrease in the gravitational acceleration \( g(z) \) with increasing height \( z \) is

\[
g(z) = \frac{r_0^2}{(r_0 + z)^2} g_0
\]

with \( r_0 = 6370 \text{ km} \) as the radius of the earth, and \( g_0 = 9.807 \text{ m/s}^2 \), the standard gravitational acceleration at sea level. With Eq. (1-8) we obtain by integration

\[
H = \int_{0}^{z} \frac{g(z)}{g_0} \, dz = \frac{z}{1 + \frac{z}{r_0}}
\]

(1-10)

For the homosphere \((z < 90 \text{ km})\), the scale height is insignificantly different from the geometric height (see Table 1-2).

The variables of state of the atmosphere can be represented by the thermal and polytropic equations of state,

\[
p = \varrho \, R 
\]

\[
\frac{p}{\varrho^n} = \text{const}
\]

(1-11a)

(1-11b)

with \( n \) as the polytropic exponent \((n < \gamma)\). From Eq. (1-11) we obtain by differentiation and elimination of \( d\varrho/dp \),

\[
\frac{dp}{\varrho} = \frac{n}{n-1} \frac{dT}{T}
\]

(1-12a)

\[
= - \frac{g_0}{RT} dH
\]

(1-12b)

The second relation follows from Eq. (1-8b). Finally, we have

\[
\frac{dT}{dH} = - \frac{n-1}{n} \frac{g_0}{R}
\]

(1-13)

Table 1-2 Reference values at the atmosphere layer boundaries*\textsuperscript{,}†

<table>
<thead>
<tr>
<th>( H_b ) [km]</th>
<th>( \varrho_b ) [km]</th>
<th>( T_b ) [K]</th>
<th>( \varrho_b ) [atm]</th>
<th>( \varrho_b ) [kg/m(^3)]</th>
<th>( \varrho_b ) [K/km]</th>
<th>( n ) [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>3.170 \times 10^{-6}</td>
<td>0</td>
<td>1</td>
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</table>

*After "U.S. Standard Atmosphere" [2].
†\( H_b, \varrho_b, T_b \) values at the lower boundary of the layer height; \( dT/dH, n \) values in the layers.
which shows that each polytropic exponent \( n \) belongs to a specific temperature gradient \( dT/dH \). Note that the gas constant* in the homosphere, up to an altitude of \( H = 90 \) km, can be taken as a constant.

From Eq. (1-13) follows by integration:

\[
T = T_b - \frac{n-1}{n} \frac{g_0}{R} (H - H_b)
\]  

(1-14)

Here it has been assumed that the polytropic exponent and, therefore, the temperature gradient are constant within a layer. The index \( b \) designates the values at the lower boundary of the layer. In Table 1-2 the values of \( H_b, z_b, T_b, \) and \( dT/dH \) are listed according to the “U.S. Standard Atmosphere” [2].

The pressure distribution with altitude of the atmosphere is obtained through integration of Eq. (1-12a) with the help of Eq. (1-14). We have

\[
\frac{P}{P_b} = \left[ 1 - \frac{n-1}{n} \frac{g_0}{RT_b} (H - H_b) \right]^{n-1}
\]

(1-15a)

For the special case \( n = 1 \) (isothermal atmosphere), Eq. (1.15a) reduces to

\[
\frac{P}{P_b} = \exp \left[ - \frac{g_0}{RT_b} (H - H_b) \right]
\]

(1-15b)

In the older literature this relationship is called the barometric height equation. Finally, the density distribution is easily found from the polytropic relation Eq. (1-11b).

Also given in Table 1-2 are the reference values \( P_b, \varrho_b \) at the layer boundaries. For the bottom layer, which reaches from sea level to \( H = 11 \) km, \( H_b = H_0 \) has to be set equal to zero in Eqs. (1-15a) and (1-15b). The other sea level values (index 0), inclusive of those for the speed of sound and the kinematic viscosity, are, after [2],

\[
\begin{align*}
g_0 &= 9.8067 \text{ m/s}^2 & t_0 &= 15^\circ \text{C} \\
p_0 &= 1.0 \text{ atm} & a_0 &= 340.29 \text{ m/s} \\
\varrho_0 &= 1.2250 \text{ kg/m}^3 & \nu_0 &= 1.4607 \times 10^{-5} \text{ m}^2/\text{s} \\
T_0 &= 288.15 \text{ K} & (dT/dH)_0 &= -6.5 \text{ K/km}
\end{align*}
\]

*The temperature gradient \( dT/dH \) determines the stability of the stratification in the stationary atmosphere. The stratification is more stable when the temperature decrease with increasing height becomes smaller. For \( dT/dH = 0 \) when \( n = 1 \), Eq. (1-13), the atmosphere is isothermal and has a very stable stratification. For \( n = \gamma = 1.405 \), the stratification is adiabatic (isentropic) with \( dT/dH = -0.98 \text{ K per 100 m} \). This stratification is indifferent, because an air volume moving upward for a certain distance cools off through expansion at just the same rate as the temperature drops with height. The air volume maintains the temperature of the ambient air and is, therefore, in an indifferent equilibrium at every altitude. Negative temperature gradients of a larger magnitude than 0.98 K/100 m result in unstable stratification.