OXFORD LECTURE SERIES IN MATHEMATICS AND ITS APPLICATIONS · 35

Flips for 3-folds and 4-folds

Edited by

Alessio Corti

Oxford Lecture Series in
Mathematics and its Applications

Series Editors
John Ball Dominic Welsh

OXFORD LECTURE SERIES IN MATHEMATICS AND ITS APPLICATIONS

Books in the series

1. I. Fonseca and W. Gangbo: *Degree theory in analysis and applications*
2. P. L. Lions: *Mathematical topics in fluid mechanics*, *Vol. 1: Incompressible models*
3. I. Anderson: *Combinatorial designs and tournaments*
4. G. David and S. W. Semmes: *Fractured fractals and broken dreams*
5. Oliver Pretzel: *Codes and algebraic curves*
6. M. Karpinski and W. Rytter: *Fast parallel algorithms for graph matching problems*
7. P. L. Lions: *Mathematical topics in fluid mechanics*, *Vol. 2: Compressible models*
8. W. T. Tutte: *Graph theory as I have known it*
9. Andrea Braides and Anneliese Defranceschi: *Homogenization of multiple integrals*
10. Thierry Cazenave and Alain Haraux: *An introduction to semilinear evolution equations*
11. J. Y. Chemin: *Perfect incompressible fluids*
12. Giuseppe Buttazzo, Mariano Giaquinta and Stefan Hildebrandt: *One-dimensional variational problems: an introduction*
13. Alexander I, Bobenko and Ruedi Seiler: *Discrete integrable geometry and physics*
14. Donia Cioranescu and Patrizia Donato: *An introduction to homogenization*
15. E. J. Janse van Rensburg: *The statistical mechanics of interacting walks*, *polygons*, *animals and vesicles*
16. S. Kuksin: *Hamiltonian partial differential equations*
17. Alberto Bressan: *Hyperbolic systems of conservation laws: the one-dimensional Cauchy problem*
18. B. Perthame: *Kinetic formulation of conservation laws*
19. A. Braides: *Gamma-convergence for beginners*
20. Robert Leese and Stephen Hurley: *Methods and Algorithms for Radio Channel Assignment*
21. Charles Semple and Mike Steel: *Phylogenetics*
22. Luigi Ambrosio and Paolo Tilli: *Topics on Analysis in Metric Spaces*
23. Eduard Feireisl: *Dynamics of Viscous Compressible Fluids*
24. Antonín Novotný and Ivan Straškraba: *Intoduction to the Mathematical Theory of Compressible Flow*
25. Pavol Hell and Jarik Nesetril: *Graphs and Homomorphisms*
26. Pavel Etingof and Frederic Latour: *The dynamical Yang-Baxter equation*, *representation theory*, *and quantum integrable systems*
27. Jorge Ramirez Alfonsin: *The Diophantine Frobenius Problem*
28. Rolf Niedermeier: *Invitation to Fixed Parameter Algorithms*
29. Jean-Yves Chemin, Benoit Desjardins, Isabelle Gallagher and Emmanuel Grenier: *Mathmatical Geophysics: An introduction to rotating fluids and the Navier-Stokes equations*
30. Juan Luis Vázquez: *Smoothing and Decay Estimates for Nonlinear Diffusion Equations*
31. Geoffrey Grimmett and Colin McDiarmid: *Combinatorics*, *Complexity and Chance*
32. Alessio Corti: *Flips for 3-folds and 4-folds*

Flips for 3-folds and 4-folds

Edited by

Alessio Corti

Department of Mathematics,
Imperial College, London

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Cape Town Dar es Salaam Hong Kong Karachi
Kuala Lumpur Madrid Melbourne Mexico City Nairobi
New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece
Guatemala Hungary Italy Japan Poland Portugal Singapore
South Korea Switzerland Thailand Turkey Ukraine Vietnam

Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries

Published in the United States
by Oxford University Press Inc., New York

First published 2007

British Library Cataloguing in Publication Data
Data available

Library of Congress Cataloging in Publication Data
Data available

Typeset by Newgen Imaging Systems (P) Ltd., Chennai, India
Printed in Great Britain
on acid-free paper by
Biddles Ltd., King's Lynn, Norfolk

ISBN 978–0–19–857061–5

1 3 5 7 9 10 8 6 4 2

Contents

This page intentionally left blank

List of Contributors

Florin Ambro
Res. Inst. Math. Sci., Kyoto University
Kyoto 606-8502, Japan

Alessio Corti
Department of Mathematics, Imperial College London
Huxley Building, 180 Queen's Gate
London SW7 2AZ, UK

Osamu Fujino
Grad. School of Math., Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan

Christopher D. Hacon
Department of Mathematics, University of Utah
155 South 1400 East
Salt Lake City UT 84112-0090, USA

János Kollár
Department of Mathematics, Princeton University
Fine Hall, Washington Road
Princeton NJ 08544-1000, USA

James M^c Kernan
Department of Mathematics, UC Santa Barbara
Santa Barbara CA 93106, USA

Hiromichi Takagi
Grad. School of Math. Sci., The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153–8914, Japan

This page intentionally left blank

1
Introduction

Alessio Corti

1.1 Minimal models of surfaces

In this book, we generalise the following theorem to 3-folds and 4-folds:

Theorem 1.1.1 (Minimal model theorem for surfaces) *Let X be a non-singular projective surface. There exists a birational morphism $f\colon X \to X'$ to a non-singular projective surface X' satisfying one of the following conditions:*

> X' **is a minimal model:** *$K_{X'}$ is nef, that is, $K_{X'} \cdot C \geq 0$ for every curve $C \subset X'$;* or
> X' **is a Mori fibre space:** *$X' \cong \mathbb{P}^2$ or X' is a $\mathbb{P}^1$-bundle over a non-singular curve T.*

This result is well known; the classical proof runs more or less as follows. If X does not satisfy the conclusion, then X contains a -1-curve, that is, a non-singular rational curve $E \subset X$ such that $K_X \cdot E = E^2 = -1$. By the Castelnuovo contractibility theorem, a -1-curve can always be contracted: there exists a morphism $f\colon E \subset X \to P \in X_1$ that maps $E \subset X$ to a non-singular point $P \in X_1$ and restricts to an isomorphism $X \smallsetminus E \to X_1 \smallsetminus \{P\}$. Viewed from $P \in X_1$, this is the blow up of $P \in X_1$. Either X_1 satisfies the conclusion, or we can continue contracting a -1-curve in X_1. Every time we contract a -1-curve, we decrease the rank of the Néron–Severi group; hence the process must terminate (stop).

1.2 Higher dimensions and flips

The conceptual framework generalising this result to higher dimensions is the well-known Mori program or minimal model program. The higher-dimensional analog $f\colon X \to X_1$ of the contraction of a -1-curve is a (extremal) *divisorial contraction*. Even if we start with X non-singular, X_1 can be singular. This is not a problem: we now know how to handle the relevant classes of singularities, for example the class of terminal singularities. The problem is that, in higher dimensions, we meet a new type of contraction:

Definition 1.2.1 A *small contraction* is a birational morphism $f\colon X \to Z$ with connected fibres such that the exceptional set $C \subset X$ is of codimension $\operatorname{codim}_X C \geq 2$.

When we meet a small contraction $f: X \to Z$, the singularities on Z are such that the first Chern class does not lift to $H^2(Z, \mathbb{Q})$: it does not make sense to form the intersection number $K_Z \cdot C$ with algebraic curves $C \subset Z$. We need a new type of operation, called a flip:

Definition 1.2.2 A small contraction $f: X \to Z$ is a *flipping contraction* if K_X is anti-ample along the fibres of f. The *flip* of f is a new small contraction $f': X' \to Z$ such that $K_{X'}$ is ample along the fibres of f'

It is conjectured that flips exist and that every sequence of flips terminates (that is, there exists no infinite sequence of flips).

1.3 The work of Shokurov

Mori [**Mor88**] first proved that flips exist if X is 3-dimensional. It was known (and, in any case, it is easy to prove) that 3-fold flips terminate; thus, Mori's result implied the minimal model theorem for 3-folds.

In his work [**Sho03**], Shokurov announced a proof of the existence of flips in dimension ≤ 4, completing a program started in [**Sho92, Kol92**]. In fact, Shokurov studies a more general kind of flip.

Definition 1.3.1 In this book, a *pair* always means a pair (X, B) where X is a normal variety, and $B = \sum b_i B_i$ is a $\mathbb{Q}$-divisor on X such that $K + B$ is $\mathbb{Q}$-Cartier.

Occasionally, B is a $\mathbb{R}$-divisor, and then $K + B$ is required to be $\mathbb{R}$-Cartier.

Let us consider a pair (X, B). For the program to work, the pair needs to have klt singularities. The precise definition is given elsewhere in this book and it is not important for the present discussion; the condition implies that all $0 < b_i < 1$ and it is satisfied, for example, if X is non-singular and the support of $B = \sum b_i B_i$ is a simple normal-crossing divisor.

Definition 1.3.2 A small contraction $f: X \to Z$ is a *klt flipping contraction* if the pair (X, B) has klt singularities and $K_X + B$ is anti-ample on the fibres of f. The *klt flip* of f (or, simply, the *flip* of f, when no confusion is likely to arise) is a new small contraction $f': X' \to Z$ such that $K_{X'} + B$ is ample on the fibres of f'

In this book, we prove the following:

Theorem 1.3.3 *The flips of klt flipping contractions $f: (X, B) \to Z$ exist if* $\dim X \leq 4$.

It is known from the work of Kawamata [**Kaw92, Kol92**] that 3-fold klt flips terminate. At the time of writing, it is not known, in full generality, that klt flips terminate in dimension ≥ 4; this book leaves the following general question open.

Problem 1.3.4 *Show that klt flips terminate in dimension* ≥ 4.

Remark 1.3.5 The recent work [**BCHM06**] settles the important special case of termination of klt flips of the *minimal model program with scaling*, *see* Section 1.21 below for a discussion of this result and some of its consequences.

1.4 Minimal models of 3-folds and 4-folds

It is known from [**KMM87**] that ordinary (that is, terminal) 4-fold flips terminate. It follows from Theorem 1.3.3 that ordinary 4-fold flips exist; thus, we have the following consequence:

Theorem 1.4.1 (Minimal model theorem for 3-folds and 4-folds) *Let X be a non-singular projective variety of dimension ≤ 4. There exists a birational map $X \dashrightarrow X'$ to a projective variety X' (with terminal singularities) satisfying one of the following conditions:*

X' **is a minimal model:** *$K_{X'}$ is nef, that is, $K_{X'} \cdot C \geq 0$ for every curve $C \subset X'$;* or

X' **is a Mori fibre space:** *There exists a morphism $\varphi\colon X' \to T$ to a variety T of smaller dimension, such that $K_{X'}$ is anti-ample on the fibres of φ. A morphism with these properties is called a* Mori fibre space.

Because 3-fold klt flips terminate [**Kaw92, Kol92**], we also have the following:

Theorem 1.4.2 (Minimal model theorem for klt 3-folds) *Let X be a non-singular projective 3-fold and $B = \sum b_i B_i$ a $\mathbb{Q}$-divisor on X, where $0 < b_i < 1$ and the support of B is a simple normal crossing divisor. There exists a birational map $f : X \dashrightarrow X'$ to a projective 3-fold X' such that the pair $(X', B' = f_* B)$ has klt singularities and satisfies one of the following conditions:*

(X', B') **is a klt minimal model:** *$K_{X'} + B'$ is nef, that is, $(K_{X'} + B') \cdot C \geq 0$ for every curve $C \subset X'$;* or

(X', B') **is a klt Mori fibre space:** *There exists a morphism $\varphi\colon X' \to T$ to a variety T of smaller dimension, such that $K_{X'} + B'$ is anti-ample on the fibres of φ. A morphism with these properties is called a* klt Mori fibre space.

If we knew that 4-fold klt flips terminated, then we could immediately generalise Theorem 1.4.2 to 4-folds.

1.5 The aim of this book

A large part of this book is a digest of the great work of Shokurov [**Sho03**]; in particular, we give a complete and essentially self-contained proof of existence of 3-fold and 4-fold klt flips.

Shokurov has introduced many new ideas in the field and has made huge progress on the conjectures on existence and termination of higher-dimensional flips. However, [**Sho03**] is very difficult to understand; in this book, we rewrite the entire subject from scratch.

Shokurov's proof of existence of 3-fold flips is conceptual. Chapter 2 on 3-fold flips aims to give a concise, complete, and pedagogical proof of the existence of 3-fold flips. I have written up the material in great detail, with the goal to make it accessible to graduate students and algebraic geometers not working in higher dimensions. I assume little prior knowledge of Mori theory; the reader who is willing to take on trust a few general results can get away with little knowledge of higher-dimensional methods.

The treatment of 4-fold flips in [**Sho03**] is much harder than that of 3-fold flips. It uses everything from the 3-fold case and much more. We tried our best to understand this argument and, after years of work, there are still a few places that we do not completely understand. Fortunately, Hacon and M^{c}Kernan [**HM05**] have found a better approach that proves a stronger theorem valid in all dimensions. Their ideas are a natural development of Shokurov's 3-dimensional proof. In Chapter 5, Hacon and M^{c}Kernan give a self-contained account of their work, showing in particular the existence of 4-fold flips.

Our main focus is to give a complete, self-contained, and mathematically solid proof of the existence of flips in dimension 3 and 4. To do this, we felt the need to rework or survey some of the foundations, particularly where results are scattered in the literature across papers by various authors often following incompatible conventions, *see* Chapters 3, 4, and 8. However, this book is not a textbook. The aim of this book is to help and stimulate research in higher-dimensional algebraic geometry. This presents some problems for the reader but also some opportunities. The material is not organically developed from first principles, as it would be natural to expect in a textbook. The payoff is that we are able to present in a coherent form some cutting-edge content in a rapidly evolving research area.

In the rest of the Introduction, I explain the main ideas of the proof of existence of 3-fold and 4-fold flips, and briefly discuss the contents of the individual chapters. I also briefly touch on the recent preprint [**BCHM06**].

1.6 Pl flips

The main result of [**Sho92**], reworked and generalised to higher dimension in [**Kol92**, Chapter 18], is a reduction of klt flips to pl flips. We review and expand the proof in Chapter 4. I briefly recall the basic definitions.

In what follows, I consider a normal variety X and a $\mathbb{Q}$-divisor $S+B$ on X. In this notation, S is a prime Weil divisor and $B=\sum b_i B_i$ is a $\mathbb{Q}$-divisor with $0<b_i<1$ having no component in common with S. The pair $(X, S+B)$ is required to have plt singularities. This notion is similar to klt singularities, but it is more general: the main difference is that the components of S appear in the boundary with coefficient 1. The precise definition is discussed later in the book and it is not crucial for understanding the outline of the proof; for example, the condition holds if X is non-singular and the support of $S+B$ is a simple normal crossing divisor.

Definition 1.6.1 A *pl flipping contraction* is a flipping contraction $f\colon X \to Z$ for the divisor $K+S+B$, such that S is anti-ample along the fibres of f. The flip of a pl flipping contraction is called a *pl flip*.

Theorem 1.6.2 *(* See [**Sho92, Kol92**] *and Chapter 4.) If n-dimensional pl flips exist and terminate, then n-dimensional klt flips exist.*

Termination of n-dimensional pl flips, also called *special termination*, is essentially a $(n-1)$-dimensional problem; this matter is treated in detail in Chapter 4.

The idea of pl flips is as follows. Because S is negative on the fibres of f, S contains all the positive dimensional fibres of f and hence the whole exceptional set. Thus, S contains information about the whole exceptional set. We may hope to reduce the existence of the flip to a problem that we can state in terms of S alone. Then we can hope to use the birational geometry of S to solve this problem. With luck we eventually construct flips by induction on $\dim X$.

1.7 b-divisors

I give a short introduction to Shokurov's notion of b-divisors. The language of b-divisors is, strictly speaking, not required for the construction of 3-fold flips in Chapter 2. Also, b-divisors are not explicitly used in the work of Hacon and M^{c}Kernan on higher-dimensional flips, but they play an important role implicitly: for instance, Lemma 5.3.19 states that an object crucial for the proof is a b-divisor. Shokurov and others use b-divisors extensively; in particular, Shokurov's important finite generation conjecture is stated in terms of b-divisors. Essentially, b-divisors are equivalent to *sheaves of fractional ideals*, and several constructions associated with the discrepancy b-divisor have a counterpart as *multiplier ideal sheaves*. Here I give a very quick introduction; more detail on b-divisors can be found in Chapter 2.

We always work with normal varieties. A *model* of a variety X is a proper birational morphism $f\colon Y \to X$ from a (normal) variety Y.

Definition 1.7.1 A *b-divisor* on X is an element:

$$\mathbf{D} \in \mathbf{Div}\, X = \lim_{Y \to X} \operatorname{Div} Y,$$

where the (projective) limit is taken over all models $f\colon Y \to X$ under the push-forward homomorphism $f_*\colon \operatorname{Div} Y \to \operatorname{Div} X$. A b-divisor $\mathbf{D}$ on X has an obvious *trace* $\mathbf{D}_Y \in \operatorname{Div} Y$ on every model $Y \to X$.

Natural constructions of divisors in algebraic geometry often give rise to b-divisors. For example, the divisor $\mathbf{div}_X\, \varphi$ of a rational function, and the divisor $\mathbf{div}_X\, \omega$ of a rational differential, are b-divisors. Indeed, if $f\colon Y \to X$ is a model, and $E \subset Y$ is a prime divisor, both $\operatorname{mult}_E \varphi$ and $\operatorname{mult}_E \omega$ are defined.

A b-divisor on X gives rise to a sheaf $\mathcal{O}_X(\mathbf{D})$ of $\mathcal{O}_X$-modules in a familiar way: if $U \subset X$ is a Zariski open subset, then

$$\mathcal{O}_X(\mathbf{D})(U) = \{\varphi \in k(X) \mid \mathbf{D}_{|U} + \mathbf{div}_U\, \varphi \geq 0\}.$$

In general, this sheaf is not quasicoherent; however, it is a coherent sheaf in all cases of interest to us. We write $H^0(X, \mathbf{D})$ for the group of global sections of $\mathcal{O}_X(\mathbf{D})$ and denote by $|\mathbf{D}| = \mathbb{P}H^0(X, \mathbf{D})$ the associated 'complete' linear system. It is crucial to understand that $H^0(X, \mathbf{D}) \subsetneq H^0(X, \mathbf{D}_X)$; the language of b-divisors is a convenient device to discuss linear systems with base conditions. Thus, b-divisors are often equivalent to *sheaves of fractional ideals*.

Example 1.7.2 The $\mathbb{Q}$-*Cartier closure* of a $\mathbb{Q}$-Cartier ($\mathbb{Q}$-)divisor D on X is the b-divisor $\overline{D}$ with trace

$$\overline{D}_Y = f^*(D)$$

on models $f\colon Y \to X$.

If $f\colon Y \to X$ is a model and D is a $\mathbb{Q}$-Cartier ($\mathbb{Q}$-)divisor on Y, we abuse notation slightly and think of $\overline{D}$ as a b-divisor on X. Indeed, f_* identifies b-divisors on Y with b-divisors on X.

Example 1.7.3 Consider a pair (X, B) of a normal variety X and $\mathbb{Q}$-divisor B; recall that we always assume that $K + B$ is $\mathbb{Q}$-Cartier. The *discrepancy* b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ is defined by:

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y$$

for all models $f\colon Y \to X$.

Definition 1.7.4(1) A b-divisor $\mathbf{D}$ on X is *b-($\mathbb{Q}$)-Cartier* if it is the Cartier closure of a ($\mathbb{Q}$)-Cartier divisor D on a model $Y \to X$.

(2) When $Y \to X$ is a model and $\mathbf{D} = \overline{\mathbf{D}_Y}$, we say that $\mathbf{D}$ *descends* to Y.

1.8 Restriction and mobile b-divisors

Definition 1.8.1 Let $\mathbf{D}$ be a b-$\mathbb{Q}$-Cartier b-divisor on X and $S \subset X$ an irreducible normal subvariety of codimension 1 not contained in the support of $\mathbf{D}_X$. I define the *restriction* $\mathbf{D}^0 = \operatorname{res}_S \mathbf{D}$ of $\mathbf{D}$ to S as follows. Pick a model $f\colon Y \to X$ such that $\mathbf{D} = \overline{\mathbf{D}_Y}$; let $S' \subset Y$ be the proper transform. I define

$$\operatorname{res}_S \mathbf{D} = \overline{\mathbf{D}_{Y|S'}},$$

where $\mathbf{D}_{Y|S'}$ is the ordinary restriction of divisors. (Strictly speaking, $\overline{\mathbf{D}_{Y|S'}}$ is a b-divisor on S'; as already noted, b-divisors on S' are canonically identified with b-divisors on S via push forward.) It is easy to see that the restriction does not depend on the choice of the model $Y \to X$.

Definition 1.8.2 An integral b-divisor $\mathbf{M}$ is *mobile* if there is a model $f\colon Y \to X$, such that

(1) $\mathbf{M} = \overline{\mathbf{M}_Y}$ is the Cartier closure of $\mathbf{M}_Y$, and
(2) the linear system (of ordinary divisors) $|\mathbf{M}_Y|$ is free on Y.

Remark 1.8.3 The restriction of a mobile b-divisor is a mobile b-divisor.

1.9 Pbd-algebras

Definition 1.9.1 A sequence $\mathbf{M}_\bullet = \{\mathbf{M}_i \mid i > 0 \text{ integer}\}$ of mobile b-divisors on X is *subadditive* if $\mathbf{M}_1 > 0$ and

$$\mathbf{M}_{i+j} \geq \mathbf{M}_i + \mathbf{M}_j$$

for all positive integers i,j. The associated *characteristic sequence* is the sequence $\mathbf{D}_i = (1/i)\mathbf{M}_i$ of b-$\mathbb{Q}$-Cartier b-divisors. We say that the characteristic sequence is *bounded* if there is a (ordinary) $\mathbb{Q}$-Cartier divisor D on X such that all $\mathbf{D}_i \leq \overline{D}$.

Remark 1.9.2 The characteristic sequence of a subadditive sequence is *convex*; that is, $\mathbf{D}_1 > 0$ and

$$\mathbf{D}_{i+j} \geq \frac{i}{i+j}\mathbf{D}_i + \frac{j}{i+j}\mathbf{D}_j \qquad \text{for all positive integers} \quad i,j.$$

Definition 1.9.3 A *pbd-algebra* is a graded algebra

$$R = R(X, \mathbf{D}_\bullet) = \bigoplus_{i\geq 0} H^0(X, i\mathbf{D}_i),$$

where $\mathbf{D}_i = (1/i)\mathbf{M}_i$ is a bounded characteristic sequence of a subadditive sequence $\mathbf{M}_\bullet$ of b-divisors.

A systematic treatment of pbd-algebras can be found in Section 2.3.45.

1.10 Restricted systems and 3-fold pl flips

Consider a pl flipping contraction $f: X \to Z$ for the divisor $K + S + B$. Let $r > 0$ be a positive integer and $D \sim r(K + S + B)$ a Cartier divisor on X; it is well known that the flip of f exists if and only if the algebra

$$R = R(X, D) = \bigoplus_{i\geq 0} H^0(X, iD)$$

is finitely generated (in fact, in that case, the flip is the Proj of this algebra). The first step is to interpret R as a suitable pbd-algebra.

Definition 1.10.1 Let D be a Cartier divisor on X. The *mobile b-part of D* is the b-divisor $\mathbf{Mob}\, D$ with trace

$$(\mathbf{Mob}\, D)_Y = \operatorname{Mob} f^*D$$

on models $f: Y \to X$, where $\operatorname{Mob} f^*D$ is the mobile part of the divisor f^*D (the part of D that moves in the linear system $|f^*D|$).

Choose, as above, a Cartier divisor $D \sim r(K + S + B)$. Denote by $\mathbf{M}_i = \mathbf{Mob}\, iD$ the mobile part and let $\mathbf{D}_i = (1/i)\mathbf{M}_i$; then, tautologically,

$$R = R(X, D) = R(X, \mathbf{D}_\bullet)$$

is a pbd-algebra. Now I come to the main idea. As I explained above, provided that S is not contained in the support of D (which is easily arranged), it makes sense to form the restriction $\mathbf{D}_i^0 = \operatorname{res}_S \mathbf{D}_i$; we consider the associated pbd-algebra on S:

$$R(S, \mathbf{D}_\bullet^0).$$

It is easy to see, though not trivial, that $R(X, \mathbf{D}_\bullet)$ is finitely generated if and only if $R(S, \mathbf{D}_\bullet^0)$ is finitely generated.

1.11 Shokurov's finite generation conjecture

In this section, I state the finite generation conjecture of Shokurov. The statement is technical; the reader may wish to just skim over it.

We want a condition on the system $\mathbf{D}^0_\bullet$ that, in appropriate situations, ensures that the pbd-algebra $R(S, \mathbf{D}^0_\bullet)$ is finitely generated. The condition is the following:

Definition 1.11.1 Let (X, B) be a pair of a variety X and divisor $B \subset X$.

(1) A b-divisor $\mathbf{M}$ on X is *canonically saturated* if there is a model $Y_1 \to X$ such that

$$\operatorname{Mob}\lceil \mathbf{M}_Y + \mathbf{A}_Y \rceil \leq \mathbf{M}_Y$$

on all models $Y \to Y_1$.

(2) A sequence $\mathbf{D}_\bullet$ of b-divisors on X is *canonically asymptotically saturated* (*canonically a-saturated*, for short) if for all i, j, there is a model $Y(i,j) \to X$ such that

$$\operatorname{Mob}\lceil (j\mathbf{D}_i + \mathbf{A})_Y \rceil \leq j\mathbf{D}_{j\,Y}$$

on all models $Y \to Y(i,j)$.

These definitions are very subtle and I will not attempt a discussion here. The reader should accept these for the moment as 'technical conditions'. A systematic treatment can be found in Sections 2.3.5 and 2.3.61.

Proposition 1.11.2 (*see* Lemma 2.3.43 and Lemma 2.4.3) *Let $(X, S + B) \to Z$ be a pl flip; the restricted system*

$$\mathbf{D}^0_\bullet = \operatorname{res}_S \mathbf{D}_\bullet$$

of Section 1.10 is canonically a-saturated.

Finite generation Conjecture 1.11.3 *Let (X, B) be a klt pair, $f : X \to Z$ a birational contraction to an affine variety Z. Assume that $K + S + B$ is anti-ample on the fibres of f. If $\mathbf{D}_\bullet$ is a canonically a-saturated convex bounded characteristic system of b-divisors on X, then the pbd-algebra $R(X, \mathbf{D}_\bullet)$ is finitely generated.*

Definition 1.11.4 An algebra satisfying the assumptions of the conjecture is called a *Shokurov algebra* (*Fano graded algebra*, or FGA algebra, in Shokurov).

A good way to get a feeling for this conjecture and the concept of canonical asymptotic saturation, is to work out the one-dimensional case; this is done in §2.3.10 and the reader may wish to read that section now. The proof of this conjecture in the case $\dim X = 2$ is a major theme in this book, *see* Theorem 2.4.10 and Corollary 7.5.2; in Chapter 9, we even prove a generalisation to non-klt surface pairs. In §2.4, the conjecture is shown assuming that $f : X \to Z$ is birational; this is sufficient for the construction of 3-fold pl flips.

It is important to realise that, in the light of the work of Hacon and M^{c}Kernan on adjoint algebras and higher-dimensional flips, *see* §1.14 below, the finite generation conjecture is no longer crucial for the construction of flips. However, the conjecture makes a deep statement about the structure of linear systems on Fano varieties, and it may have a role to play in future work on finite generation.

1.12 What is log terminal?

Fujino's Chapter 3 is an essay on the definition of log terminal singularities of pairs. The category of pairs (X, B) of a variety X and a divisor $B \subset X$ was first introduced by Iitaka and his group. In the early days, $B = \sum B_i$ was a reduced integral divisor and one was really interested in the non-compact variety $U = X \setminus B$. It is an easy consequence of Hironaka's resolution theorem that, if X is non-singular and B is a simple normal crossing divisor, then the *log plurigenera* $h^0\big(X, n(K_X + B)\big)$ depend only on U and not on the choice of the compactification X and boundary divisor B. This suggests that it should be possible to generalise birational geometry, minimal models, etc. to non-compact varieties, or rather *pairs* (X, B) of a variety X and *boundary* divisor $B \subset X$. As in the absolute case, it is necessary to allow some singularities. There should be a notion of *log terminal singularities of pairs*, corresponding to terminal singularities of varieties. To define such a notion turned out to be a very subtle technical problem. Many slightly different inequivalent definitions were proposed; for example [**Kol92**] alone contains a dozen variants. Over the years, we believe, one particular notion, called *divisorially log terminal* pairs in [**KM98**], has proved itself to be the most useful. In this book, we work exclusively with divisorially log terminal (abbreviated *dlt*) pairs. The book [**KM98**] contains a clear and technically precise exposition of divisorially log terminal singularities and we adopt it as our main reference. However, the professional in higher-dimensional geometry must be able to read the literature; in particular, a good knowledge of at least the fundamental texts [**KMM87, Sho92, Kol92**] is essential. There is no agreement on the basic definitions among these texts. This state of affairs creates serious difficulties for everybody interested in these things. The chapter by Fujino is a guide to the different definitions existing in the literature; it is the only such guide known to me. In it the reader will find a discussion of the various definitions, their properties and respective merits and the state of the art on the various implications existing among them, as well as several illustrative examples.

In particular, the chapter contains a careful treatment of the adjunction formula for dlt pairs. If X is a normal variety and $S \subset X$ a reduced Cartier divisor, the adjunction formula states that $K_S = K_X + S_{|S}$. The adjunction formula is the most important tool that allows us to manipulate the canonical divisor. If S is reduced of codimension 1, but not necessarily a Cartier divisor, under mild assumptions, there is a naturally defined effective $\mathbb{Q}$-divisor $\mathrm{Diff} > 0$ on S such that the adjunction formula

$$K_S + \mathrm{Diff} = (K_X + S)_{|S}$$

holds. For example, consider the case of a quadric cone $X \subset \mathbb{P}^3$ and a line $S \subset X$; the line must pass through the vertex $P \in S$ of the cone, and $\mathrm{Diff} = (1/2)P$. More generally, consider a pair $(X, S + B)$, where B is a strict boundary and S is a reduced divisor not contained in $\mathrm{Supp}\, B$. Under fairly mild conditions, one can define a different $\mathrm{Diff}\,(B) > 0$ and $K_S + \mathrm{Diff}\,(B) = (K_X + S + B)_{|S}$. It is not difficult to show that: If $(X, S + B)$ is plt,