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Fourier–Mukai transforms in algebraic geometry

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PREFACE

This book is based on a course given at the Institut de Mathématiques de Jussieu in 2004 and once more in 2005. It was conceived as a first specialized course in algebraic geometry. A student with a basic knowledge in algebraic geometry, e.g. a student having worked through the first three chapters of Hartshorne's book [45], should be able to follow the lectures without much trouble. Occasionally, notions from other areas, e.g. singular cohomology, Hodge theory, abelian varieties, K3 surfaces, were needed, which were then presented in a rather ad hoc manner, tailor-made for the purposes of the course. With a few exceptions full proofs are given. The exercises included in the text should help the reader to gain a working knowledge of the subject.

What is this book about? Its principal character is the derived category of coherent sheaves on a smooth projective variety. Derived categories of this type have been known for many years. Although widely accepted as the right framework for any kind of derived functors, e.g. cohomology groups, higher direct images, etc., they were usually considered as rather formal objects without much interesting internal structure. Contrary to the cohomology and the Chow ring of a projective variety X, the derived category of coherent sheaves as an invariant of X had not been investigated thoroughly. This has changed drastically over the last ten years.

The origin of the theory as treated here however goes back to celebrated papers by Mukai, more than twenty years ago. He constructed geometrically motivated equivalences between derived categories of non-isomorphic varieties. Also, over many years the Moscow school had constantly worked on the description of coherent sheaves on homogenous varieties, e.g. the projective space, Grassmannians, etc. On the other hand, Kontsevich's homological mirror symmetry has revived the interest in these questions outside the small circle of experts. Roughly, Kontsevich proposed to view mirror symmetry as an equivalence of the derived category of coherent sheaves of certain projective varieties with the Fukaya category associated to the symplectic geometry of the mirror variety. Although we deliberately do not enter into the details of this relation, it is this point of view that motivates and in some sense explains many of the central results as well as open problems in this area.

The derived category turns out to be a very reasonable invariant. Due to results of Bondal and Orlov one knows that it determines the variety whenever the canonical bundle is either ample or anti-ample. If this was true without any assumptions on the positivity of the canonical bundle, the theory would be without much interest. However, there is a region in the classification of projective varieties where the derived category turns out to be less rigid without getting completely out of hand. The most prominent example was observed by Mukai in the very first paper on the subject. He showed that the Poincaré bundle induces an equivalence between the derived category of an abelian variety A and the derived category of its dual $\widehat{A}$ (which in general is not isomorphic to A). These results, to be discussed in detail in various chapters, naturally lead to the question under which conditions two smooth projective varieties give rise to equivalent derived categories. This is the central theme of this book.

One word on the choice of the material. Everything that did not have a distinctive geometric touch has been left out. In particular, questions related to representation theory, e.g. of quivers, or to modules over (non-commutative) rings, have not been touched upon. This choice is due to personal taste, limitations by a one semester course and my own ignorance in some of these areas.

We refrain from giving a lengthy introduction to the contents of every chapter. A glance at the table of contents will give a first impression of which topics are treated, and the remarks at the beginning of each chapter provide more details. The reader familiar with the general yoga of derived categories and derived functors may go directly to Chapter 4 or 5 and come back to some of the background material collected in the first three chapters whenever needed.

Acknowledgements: I am intellectually indebted to A. Bondal, T. Bridgeland, Y. Kawamata, S. Mukai, and D. Orlov. The overwhelming part of the theory as presented here is due to them. The idea that this text could help to stimulate newcomers to pursue research originated by them was the driving force during the preparation of these notes.

I am particularly grateful to the Institut de Mathématiques de Jussieu for giving me (twice) the opportunity to teach the course this book is based on. The intellectual atmosphere at the institute has been very stimulating throughout the whole project and I have fond memories of all the discussions I had with my colleagues at the IJM during this time. In particular, I wish to thank J. Le Potier and R. Rouquier.

Many people have made comments on earlier versions of the book. I wish to thank U. Görtz, M. Nieper-Wißkirchen, K. Oguiso, D. Ploog, P. Stellari, R. Thomas, and D. Ben-Zvi. I am most grateful to Bahoua Fu and Justin Sawon who carefully worked through the book and sent me detailed lists of corrections, questions, and remarks.

CONTENTS

1

TRIANGULATED CATEGORIES

The reader familiar with the basic notions of abelian and derived categories may only need to browse through this section or skip it altogether. It will be much more interesting to come back to the specific results discussed here when, in the later chapters, they are actually applied to geometrically concrete problems. However, the reader not feeling completely at ease with the formal language of category theory should work through this chapter in order to be well prepared for everything that follows.

We hope that separating results from category theory from the other chapters rather than blending them in later when used, will help readers to understand which part of the theory is really geometrical and which is more formal.

On the other hand, this chapter is not meant as a thorough introduction to the subject. We only present those parts of the theory that are relevant in our context.

We will not worry about any kind of set theoretical issues and will always assume we remain in a given universe (or, as put in [39, p.58], 'that all the required hygiene regulations are obeyed').

1.1 Additive categories and functors

We suppose that the reader is familiar with the notion of a category and of a functor between two categories. For the reader's convenience we briefly recall a few central notions. If not otherwise stated all functors are covariant.

Definition 1.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is* full *if for any two objects $A, B \in \mathcal{A}$ the induced map*

$$F : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(F(A), F(B))$$

is surjective. The functor F is called faithful *if this map is injective for all $A, B \in \mathcal{A}$.*

A *morphism* $F \longrightarrow F'$ between two functors $F, F' : \mathcal{A} \longrightarrow \mathcal{B}$ is given by morphisms $\varphi_A \in \mathrm{Hom}(F(A), F'(A))$ for any object $A \in \mathcal{A}$ which are functorial in A, i.e. $F'(f) \circ \varphi_A = \varphi_B \circ F(f)$ for any $f : A \longrightarrow B$.

Definition 1.2 *Two functors $F, F' : \mathcal{A} \longrightarrow \mathcal{B}$ are* isomorphic *if there exists a morphism of functors $\varphi : F \longrightarrow F'$ such that for any object $A \in \mathcal{A}$ the induced morphism $\varphi_A : F(A) \longrightarrow F'(A)$ is an isomorphism (in $\mathcal{B}$).*

Equivalently, F and F' are isomorphic if there exist functor morphisms $\varphi : F \longrightarrow F'$ and $\psi : F' \longrightarrow F$ with $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$.

Definition 1.3 *A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called an* equivalence *if there exists a functor $F^{-1} : \mathcal{B} \longrightarrow \mathcal{A}$ such that $F \circ F^{-1}$ is isomorphic to $\mathrm{id}_{\mathcal{B}}$ and $F^{-1} \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{A}}$. One calls F^{-1} an* inverse *or, sometimes,* quasi-inverse *of F.*

Two categories $\mathcal{A}$ and $\mathcal{B}$ are called equivalent *if there exists an equivalence $F : \mathcal{A} \longrightarrow \mathcal{B}$.*

Clearly, any equivalence is fully faithful. A partial converse is provided by

Proposition 1.4 *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a fully faithful functor. Then F is an equivalence if and only if every object $B \in \mathcal{B}$ is isomorphic to an object of the form $F(A)$ for some $A \in \mathcal{A}$.*

Proof In order to define the inverse functor F^{-1}, one chooses for any $B \in \mathcal{B}$ an object $A_B \in \mathcal{A}$ together with an isomorphism $\varphi_B : F(A_B) \xrightarrow{\sim} B$. Then, let

$$F^{-1} : \mathcal{B} \longrightarrow \mathcal{A}$$

be the functor that associates to any object $B \in \mathcal{B}$ this distinguished object $A_B \in \mathcal{A}$ and for which $F^{-1} : \mathrm{Hom}(B_1, B_2) \longrightarrow \mathrm{Hom}(F^{-1}(B_1), F^{-1}(B_2))$ is given by the composition of

$$\mathrm{Hom}(B_1, B_2) \xrightarrow{\sim} \mathrm{Hom}(F(A_{B_1}), F(A_{B_2})), \quad f \longmapsto \varphi_{B_2}^{-1} \circ f \circ \varphi_{B_1}$$

and the inverse of the bijection

$$F : \mathrm{Hom}(A_{B_1}, A_{B_2}) \xrightarrow{\sim} \mathrm{Hom}(F(A_{B_1}), F(A_{B_2})).$$

The isomorphisms $F \circ F^{-1} \simeq \mathrm{id}_{\mathcal{B}}$ and $F^{-1} \circ F \simeq \mathrm{id}_{\mathcal{A}}$ are the ones that are naturally induced by the isomorphisms φ_B. □

The proposition immediately yields the

Corollary 1.5 *Any fully faithful functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ defines an equivalence between $\mathcal{A}$ and the full subcategory of $\mathcal{B}$ of all objects $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$.* □

In the following proposition we let $\mathrm{Fun}(\mathcal{A})$ be the category of all contravariant functors, i.e. the objects are functors $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and the morphisms are functor morphisms. Consider the natural functor

$$\mathcal{A} \longrightarrow \mathrm{Fun}(\mathcal{A}), \qquad A \longmapsto \mathrm{Hom}(\ \ , A)$$

Proposition 1.6 **(Yoneda lemma)** *This functor $\mathcal{A} \longrightarrow \mathrm{Fun}(\mathcal{A})$ defines an equivalence of $\mathcal{A}$ with the full subcategory of* representable functors *F, i.e. functors isomorphic to some $\mathrm{Hom}(\ \ , A)$. In particular, $A \longmapsto \mathrm{Hom}(\ \ , A)$ is fully faithful.*

Proof See [39, II.3]. □

We will rarely work with completely arbitrary categories. All our categories will at least be additive.

Definition 1.7 *A category* $\mathcal{A}$ *is an* additive category *if for every two objects* $A, B \in \mathcal{A}$ *the set* $\mathrm{Hom}(A, B)$ *is endowed with the structure of an abelian group such that the following three conditions are satisfied:*

i) *The compositions* $\mathrm{Hom}(A_1, A_2) \times \mathrm{Hom}(A_2, A_3) \longrightarrow \mathrm{Hom}(A_1, A_3)$ *written as* $(f, g) \longmapsto g \circ f$ *are bilinear.*

ii) *There exists a* zero object $0 \in \mathcal{A}$*, i.e. an object* 0 *such that* $\mathrm{Hom}(0, 0)$ *is the trivial group with one element.*

iii) *For any two objects* $A_1, A_2 \in \mathcal{A}$ *there exists an object* $B \in \mathcal{A}$ *with morphisms* $j_i : A_i \longrightarrow B$ *and* $p_i : B \longrightarrow A_i$*,* $i = 1, 2$*, which make* B *the* direct sum *and the* direct product *of* A_1 *and* A_2*.*

We tacitly assume the usual compatibilities $p_i \circ j_i = \mathrm{id}$, $p_2 \circ j_1 = p_1 \circ j_2 = 0$, and $j_1 \circ p_1 + j_2 \circ p_2 = \mathrm{id}$, which hold automatically up to automorphisms of B.

Exercise 1.8 Show that for any object $A \in \mathcal{A}$ in an additive category $\mathcal{A}$ there exist unique morphisms $0 \longrightarrow A$ and $A \longrightarrow 0$. The existence of such an object 0 in a category $\mathcal{A}$ is of course equivalent to ii).

A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between additive categories $\mathcal{A}$ and $\mathcal{B}$ will usually be assumed to be *additive*, i.e. the induced maps $\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(F(A), F(B))$ are group homomorphisms.

Everything that has been said so far carries over to additive categories. In particular, an additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ which is an equivalence is in fact an additive equivalence, i.e. the inverse functor F^{-1} is additive as well. The Yoneda lemma is modified as follows: For an additive category $\mathcal{A}$ we let $\mathrm{Fun}(\mathcal{A})$ be the category of contravariant additive(!) functors $F : \mathcal{A} \longrightarrow \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. Then the Yoneda lemma in the form of Proposition 1.6 remains valid.

We will go one step further. As the categories we will eventually be interested in have geometric origin, i.e. are defined in terms of certain varieties over some base field, we usually deal with the following special type of additive categories. In the following we denote by k an arbitrary field.

Definition 1.9 *A* k-linear category *is an additive category* $\mathcal{A}$ *such that the groups* $\mathrm{Hom}(A, B)$ *are* k*-vector spaces and such that all compositions are* k*-bilinear.*

Additive functors between two k*-linear additive categories over a common base field* k *will be assumed to be* k-linear*, i.e. for any two objects* $A, B \in \mathcal{A}$ *the map* $F : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(F(A), F(B))$ *is* k*-linear.*

Once again, everything that has been mentioned before carries over literally to additive categories over a field. Usually we will state all abstract results for additive categories, but in the applications everything will be over a base field. In principle, though, it could happen that two k-linear categories are equivalent as ordinary additive categories without being equivalent as k-linear categories.

The Yoneda lemma can again be adjusted to the situation: this time, one considers the category of contravariant k-linear functors from $\mathcal{A}$ into the category $\mathbf{Vec}(k)$ of k-vector spaces.

Definition 1.10 *An additive category $\mathcal{A}$ is called* abelian *if also the following condition holds true:*

iv) *Every morphism $f \in \mathrm{Hom}(A, B)$ admits a kernel and a cokernel and the natural map $\mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$ is an isomorphism.*

Recall that the image $\mathrm{Im}(f)$ is a kernel for a cokernel $B \longrightarrow \mathrm{Coker}(f)$ and the coimage $\mathrm{Coim}(f)$ is a cokernel for a kernel $\mathrm{Ker}(f) \longrightarrow A$. So, condition iv) says that for any $f : A \longrightarrow B$ there exists the following diagram

$$\begin{array}{ccccccc} \mathrm{Ker}(f) & \xrightarrow{i} & A & \xrightarrow{\quad\quad f \quad\quad} & B & \xrightarrow{\pi} & \mathrm{Coker}(f). \\ & & & \searrow \qquad\qquad \nearrow & & & \\ & & & \mathrm{Coker}(i) \xrightarrow{\sim} \mathrm{Ker}(\pi) & & & \end{array}$$

In particular, the notion of exact sequences is usually only considered for abelian categories. We recall that a sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

is called *exact* if and only if $\mathrm{Ker}(f_2) = \mathrm{Im}(f_1)$.

Examples 1.11 i) Let R be a commutative ring. Then the category $\mathbf{Mod}(R)$ of R-modules is abelian. The full subcategory of finitely generated modules is abelian as well.

ii) Let X be a topological space. Then the category of sheaves of abelian groups $\mathbf{Sh}(X)$ is abelian. If a sheaf of commutative rings on X is fixed, then the subcategory of sheaves of modules over this sheaf of rings is again abelian.

iii) Let X be a scheme. Then the categories $\mathbf{Coh}(X)$ and $\mathbf{Qcoh}(X)$ of all coherent respectively quasi-coherent sheaves on X are both abelian.

Suppose $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor between abelian categories. In particular, any sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

with $f_2 \circ f_1 = 0$ (or, in other words, $\mathrm{Im}(f_1) \subset \mathrm{Ker}(f_2)$) is mapped to

$$F(A_1) \xrightarrow{F(f_1)} F(A_2) \xrightarrow{F(f_2)} F(A_3)$$

again with $F(f_2) \circ F(f_1) = F(f_2 \circ f_1) = 0$.

Definition 1.12 *The functor F is* left (right) exact *if any short exact sequence*

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

is mapped to a sequence

$$0 \longrightarrow F(A_1) \xrightarrow{F(f_1)} F(A_2) \xrightarrow{F(f_2)} F(A_3) \longrightarrow 0$$

which is exact except possibly in $F(A_3)$ (respectively in $F(A_1)$). The functor is exact *if it is left and right exact.*

Exercise 1.13 Show that a functor F is left exact if and only if any exact sequence $0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3$ (no surjectivity on the right!) induces an exact sequence $0 \longrightarrow F(A_1) \longrightarrow F(A_2) \longrightarrow F(A_3)$.

Examples 1.14 i) Let $\mathcal{A}$ be an abelian category and $A_0 \in \mathcal{A}$. Then

$$\mathrm{Hom}(A_0, \quad) : \mathcal{A} \longrightarrow \mathbf{Ab}$$

is a left exact functor. The contravariant functor

$$\mathrm{Hom}(\quad, A_0) : \mathcal{A} \longrightarrow \mathbf{Ab}$$

is also left exact. (Left exactness of a contravariant functor $F : \mathcal{A} \longrightarrow \mathbf{Ab}$ means by definition left exactness of the covariant functor $F : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$.)

ii) Recall that an object $P \in \mathcal{A}$ is called *projective* if $\mathrm{Hom}(P, \quad)$ is right exact (and hence exact). An object $I \in \mathcal{A}$ is called *injective* if $\mathrm{Hom}(\quad, I)$ is right exact (and hence exact).

iii) Free modules over a ring R are projective objects in $\mathbf{Mod}(R)$. But (locally) free sheaves in $\mathbf{Coh}(X)$ are almost never projective.

Definition 1.15 *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between arbitrary categories.*

A functor $H : \mathcal{B} \longrightarrow \mathcal{A}$ is right adjoint *to F (one writes $F \dashv H$) if there exist isomorphisms*

$$\mathrm{Hom}(F(A), B) \simeq \mathrm{Hom}(A, H(B)) \tag{1.1}$$

for any two objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ which are functorial in A and B.

A functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ is left adjoint *to F (one writes $G \dashv F$) if there exist isomorphisms* $\mathrm{Hom}(B, F(A)) = \mathrm{Hom}(G(B), A)$ *for any two objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ which are functorial in A and B.*

Clearly, H is right adjoint to F if and only if F is left adjoint to H.

Remarks 1.16 i) Suppose $F \dashv H$. Then $\mathrm{id}_{F(A)} \in \mathrm{Hom}(F(A), F(A))$ induces a morphism $A \longrightarrow H(F(A))$. The naturality of isomorphisms in the definition of the adjoint functor ensures that these morphisms define a functor morphism

$$h : \mathrm{id}_{\mathcal{A}} \longrightarrow H \circ F.$$

In the same vein, inserting $A = H(B)$ in (1.1) yields a canonical morphism $F(H(B)) \to B$ and, therefore, a functor morphism

$$g : F \circ H \longrightarrow \mathrm{id}_{\mathcal{B}}.$$

ii) Using the Yoneda lemma 1.6, one verifies that a left (or right) adjoint functor, if it exists at all, is unique up to isomorphism. More explicitly, for two right adjoint functors H and H' of F one defines an isomorphism $H \simeq H'$ which for any $B \in \mathcal{B}$ is given as the image of the identity under the functorial isomorphism $\mathrm{Hom}(H(B), H(B)) \simeq \mathrm{Hom}(F(H(B)), B) \simeq \mathrm{Hom}(H(B), H'(B))$.

iii) If F is an additive functor (in particular, $\mathcal{A}$ and $\mathcal{B}$ are additive), then one requires the isomorphisms (1.1) to be isomorphisms of abelian groups. Similar, if everything is k-linear, then also these isomorphisms are required to be k-linear. A priori, one cannot exclude the pathological case of an adjoint functor that is not additive, although the functor itself is. This can only occur if the isomorphism in (1.1) is not a group homomorphism.

iv) If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is left adjoint to $H : \mathcal{B} \to \mathcal{A}$, then F is right exact and H is left exact. Note that even when F is left and right exact, its right adjoint is in general only left exact.

Exercise 1.17 Suppose $F \dashv H$. Show that

$$f \longmapsto \left(A \xrightarrow{h_A} H(F(A)) \xrightarrow{H(f)} H(B)\right)$$

describes the adjunction morphism $\mathrm{Hom}(F(A), B) = \mathrm{Hom}(A, H(B))$.

Exercise 1.18 Prove assertion iv) above.

Exercise 1.19 Suppose $F \dashv H$. Show that for the induced morphisms $g : F \circ H \to \mathrm{id}$ and $h : \mathrm{id} \to H \circ F$ the composition

$$H \xrightarrow{h_{H(\ )}} (H \circ F) \circ H = H \circ (F \circ H) \xrightarrow{H(g)} H$$

is the identity. See [72, IV.1] and [39, II.3] for a converse.

Examples 1.20 Let $f : X \to Y$ be a morphism between two noetherian schemes X and Y. Then the pull-back functor

$$f^* : \mathbf{Qcoh}(Y) \longrightarrow \mathbf{Qcoh}(X)$$

is right exact and taking the direct image

$$f_* : \mathbf{Qcoh}(X) \longrightarrow \mathbf{Qcoh}(Y)$$

is left exact. Moreover, $f^* \dashv f_*$. If f is proper, the same holds for the categories of coherent sheaves on X and Y.

Lemma 1.21 *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and $G \dashv F$. Then the induced functor morphism $g : G \circ F \longrightarrow \mathrm{id}_{\mathcal{A}}$ induces for any $A, B \in \mathcal{A}$ the following commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & & \\ \downarrow{\scriptstyle \circ g_A} & \searrow^{F} & \\ \mathrm{Hom}(G(F(A)),B) & \xrightarrow{\sim} & \mathrm{Hom}(F(A),F(B)). \end{array}$$

Here, the isomorphism is given by adjunction.

Similarly, if $F \dashv H$ then the natural functor morphism $h : \mathrm{id}_{\mathcal{A}} \longrightarrow H \circ F$ induces for all $A, B \in \mathcal{A}$ the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & \xrightarrow{h_B \circ} & \mathrm{Hom}(A,H(F(B))) \\ & \searrow_{F} & \downarrow{\scriptstyle \wr} \\ & & \mathrm{Hom}(F(A),F(B)). \end{array}$$

Again, the isomorphism is given by adjunction.

Proof As $G \dashv F$, the following diagram commutes for all $f : A \longrightarrow B$ and all $C \in \mathcal{B}$:

$$\begin{array}{ccc} \mathrm{Hom}(G(C),A) & \xrightarrow{\sim} & \mathrm{Hom}(C,F(A)) \\ {\scriptstyle f\circ}\downarrow & \circlearrowleft & \downarrow{\scriptstyle F(f)\circ} \\ \mathrm{Hom}(G(C),B) & \xrightarrow{\sim} & \mathrm{Hom}(C,F(B)). \end{array}$$

Applied to $C = F(A)$ it yields

$$\begin{array}{ccc} \mathrm{Hom}(G(F(A)),A) & \xrightarrow{\sim} & \mathrm{Hom}(F(A),F(A)) \\ \downarrow & \circlearrowleft & \downarrow \\ \mathrm{Hom}(G(F(A)),B) & \xrightarrow{\sim} & \mathrm{Hom}(F(A),F(B)). \end{array}$$

Clearly, the vertical homomorphism on the right sends $\mathrm{id}_{F(A)}$ to $F(f)$. On the other hand, its image under

$$\mathrm{Hom}(F(A),F(A)) \simeq \mathrm{Hom}(G(F(A)),A) \longrightarrow \mathrm{Hom}(G(F(A)),B)$$

is just $f \circ g_A$.

This proves the commutativity of the lower triangle. The commutativity of the upper one is proved similarly. □

Corollary 1.22 *Suppose a fully faithful functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ admits a left adjoint $G \dashv F$. Then the natural functor morphism*

$$g : G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathcal{A}}$$

is an isomorphism.

Similarly, if a fully faithful functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ admits a right adjoint $F \dashv H$, then the natural functor morphism

$$h : \mathrm{id}_{\mathcal{A}} \xrightarrow{\sim} H \circ F$$

is an isomorphism.

Proof Since $F : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(F(A), F(B))$ is bijective, the commutativity of the diagram above proves that $G \circ F \longrightarrow \mathrm{id}_{\mathcal{A}}$ induces bijections

$$\mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}((G \circ F)(A), B)$$

for all A and B. By the Yoneda lemma 1.6, this shows that $G \circ F \longrightarrow \mathrm{id}_{\mathcal{A}}$ is an isomorphism. The proof of the second statement is similar. □

The same arguments also show the converse:

Corollary 1.23 *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be two functors such that $G \dashv F$. If the induced functor morphism $G \circ F \longrightarrow \mathrm{id}_{\mathcal{A}}$ is an isomorphism, then F is fully faithful.*

Similarly, if $F \dashv H$ such that $\mathrm{id}_{\mathcal{A}} \longrightarrow H \circ F$ is an isomorphism, then F is fully faithful. □

Remark 1.24 In short, if $F \dashv H$, then:

$$F \text{ is fully faithful} \iff h : \mathrm{id}_{\mathcal{A}} \xrightarrow{\sim} H \circ F$$

and if $G \dashv F$, then:

$$F \text{ is fully faithful} \iff g : G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathcal{A}}.$$

Exercise 1.25 Suppose $G \dashv F \dashv H$ and F fully faithful. Construct a canonical homomorphism $H \longrightarrow G$.

In many cases, adjoint functors exist. The case that interests us most is the case of equivalences. Here, the existence of left and right adjoints is granted by the following general result.

Proposition 1.26 *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an equivalence of categories. Then F admits a left adjoint and a right adjoint. More precisely, if $F' : \mathcal{B} \longrightarrow \mathcal{A}$ is an inverse functor of F then $F \dashv F' \dashv F$.*

Proof Very roughly, this is due to the following sequence of functorial isomorphisms

$$\mathrm{Hom}(F(A), B) \simeq \mathrm{Hom}(F'(F(A)), F'(B)) \simeq \mathrm{Hom}(A, F'(B)),$$

where we use $F'(F(A)) \simeq A$. Details are left to the diligent reader. □

Remark 1.27 These results justify the approach that is usually followed when proving the equivalence of certain categories: Suppose F is a functor that is hoped to be an equivalence and that admits a left adjoint $G \dashv F$ (or right adjoint $F \dashv H$). Then one checks whether the adjunction morphism $G \circ F \longrightarrow \mathrm{id}$ (respectively $\mathrm{id} \longrightarrow H \circ F$) is bijective. If so, the functor F is fully faithful. Eventually, one has to ensure that any object in the target category is isomorphic to an object in the image of F.

Definition 1.28 *Let $\mathcal{A}$ be a k-linear category. A* Serre functor *is a k-linear equivalence $S : \mathcal{A} \longrightarrow \mathcal{A}$ such that for any two objects $A, B \in \mathcal{A}$ there exists an isomorphism*

$$\eta_{A,B} : \mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}(B, S(A))^*$$

(of k-vector spaces) which is functorial in A and B.

We write the induced pairing as

$$\mathrm{Hom}(B, S(A)) \times \mathrm{Hom}(A, B) \longrightarrow k \ , \ (f, g) \longmapsto \langle f|g\rangle.$$

Remark 1.29 In the original paper by Bondal and Kapranov [13] an additional condition was required, namely that for any two objects $A, B \in \mathcal{A}$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(A, B) & \xrightarrow{\eta_{A,B}} & \mathrm{Hom}(B, S(A))^* \\ \downarrow S & \circlearrowleft & \uparrow S^* \\ \mathrm{Hom}(S(A), S(B)) & \xrightarrow{\eta_{S(A),S(B)}} & \mathrm{Hom}(S(B), S^2(A))^*. \end{array}$$

It turns out that this is automatically satisfied.[1] Indeed, inserting the additional diagonal arrow $\eta^*_{B,S(A)} : \mathrm{Hom}(S(A), S(B)) \longrightarrow \mathrm{Hom}(B, S(A))^*$ induced by the

[1] Thanks to Raphael Rouquier for explaining this to me.

defining property of a Serre functor, one reduces to the commutativity of the two triangles. More precisely, what we denote by $\eta^*_{B,S(A)}$ is in fact the composition of $\mathrm{Hom}(S(A),S(B)) \longrightarrow \mathrm{Hom}(S(A),S(B))^{**}$ with the actual $\eta^*_{B,S(A)}$. Thus one has to show that

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & & \\ & \searrow^{\eta_{A,B}} & \\ {\scriptstyle S}\downarrow & & \mathrm{Hom}(B,S(A))^* \\ & \nearrow_{\eta^*_{B,S(A)}} & \\ \mathrm{Hom}(S(A),S(B)) & & \end{array}$$

is commutative or, equivalently, that for $f \in \mathrm{Hom}(B,S(A))$ and $g \in \mathrm{Hom}(A,B)$ one has $\langle f|g\rangle = \langle S(g)|f\rangle$. Since η is functorial in the second variable, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & \xrightarrow{\eta_{A,B}} & \mathrm{Hom}(B,S(A))^* \\ {\scriptstyle \circ g}\uparrow & \circlearrowleft & \uparrow{\scriptstyle (S(g)\circ\ )^*} \\ \mathrm{Hom}(B,B) & \xrightarrow{\eta_{B,B}} & \mathrm{Hom}(B,S(B))^*. \end{array}$$

Applied to id $\in \mathrm{Hom}(B,B)$ it yields $\langle f|g\rangle = \langle (S(g)\circ f)|\mathrm{id}\rangle$. We next claim that $\langle (S(g)\circ f)|\mathrm{id}\rangle = \langle S(g)|f\rangle$, which can be seen by commutativity of the analogous diagram (which uses functoriality of η in the first variable)

$$\begin{array}{ccc} \mathrm{Hom}(B,B) & \xrightarrow{\eta_{B,B}} & \mathrm{Hom}(B,S(B))^* \\ {\scriptstyle f\circ}\downarrow & \circlearrowleft & \downarrow{\scriptstyle (\ \circ f)^*} \\ \mathrm{Hom}(B,S(A)) & \xrightarrow{\eta_{B,S(A)}} & \mathrm{Hom}(S(A),S(B))^*. \end{array}$$

In order to avoid any trouble with the dual, one usually assumes that all Hom's in $\mathcal{A}$ are finite-dimensional. Under this hypothesis it is easy to see that a Serre functor, if it exists, is unique up to isomorphism. More generally one has the following

Lemma 1.30 *Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear categories over a field k with finite-dimensional Hom's. If $\mathcal{A}$ and $\mathcal{B}$ are endowed with a Serre functor $S_{\mathcal{A}}$, respectively $S_{\mathcal{B}}$, then any k-linear equivalence*

$$F : \mathcal{A} \longrightarrow \mathcal{B}$$

commutes with Serre duality, i.e. there exists an isomorphism

$$F \circ S_{\mathcal{A}} \simeq S_{\mathcal{B}} \circ F.$$

Proof This is an application of the Yoneda lemma 1.6: since F is fully faithful, one has for any two objects $A, B \in \mathcal{A}$

$$\mathrm{Hom}(A, S(B)) \simeq \mathrm{Hom}(F(A), F(S(B))) \text{ and } \mathrm{Hom}(B, A) \simeq \mathrm{Hom}(F(B), F(A)).$$

Together with the two isomorphisms

$$\mathrm{Hom}(A, S(B)) \simeq \mathrm{Hom}(B, A)^* \text{ and } \mathrm{Hom}(F(B), F(A)) \simeq \mathrm{Hom}(F(A), S(F(B)))^*,$$

this yields a functorial (in A and B) isomorphism

$$\mathrm{Hom}(F(A), F(S(B))) \simeq \mathrm{Hom}(F(A), S(F(B))).$$

Using the hypothesis that F is an equivalence and, in particular, that any object in $\mathcal{B}$ is isomorphic to some $F(A)$, one concludes that there exists a functor isomorphism $F \circ S_{\mathcal{A}} \simeq S_{\mathcal{B}} \circ F$. □

Remark 1.31 Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between k-linear categories $\mathcal{A}$ and $\mathcal{B}$ endowed with Serre functors $S_{\mathcal{A}}$, respectively $S_{\mathcal{B}}$. Then

$$G \dashv F \quad \Rightarrow F \dashv S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1}.$$

(As before we assume that all Hom's are finite-dimensional.)

Indeed, under the given assumptions we have the following functorial isomorphisms:

$$\begin{aligned} \mathrm{Hom}(A_1, (S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1})(A_2)) &\simeq \mathrm{Hom}((G \circ S_{\mathcal{B}}^{-1})(A_2), A_1)^* \\ &\simeq \mathrm{Hom}(S_{\mathcal{B}}^{-1}(A_2), F(A_1))^* \\ &\simeq \mathrm{Hom}(F(A_1), S_{\mathcal{B}}(S_{\mathcal{B}}^{-1}(A_2))) \\ &\simeq \mathrm{Hom}(F(A_1), A_2). \end{aligned}$$

A similar argument allows the construction of a left adjoint if a right adjoint $F \dashv H$ is given. In particular, for functors between categories with Serre functors the existence of the left or the right adjoint implies the existence of the other one.

1.2 Triangulated categories and exact functors

Triangulated categories, the kind of categories we will be interested in throughout, were introduced independently and around the same time by Puppe [99] and in Verdier's thesis [118] under the supervision of Grothendieck. We recommend [39, 61, 88] for a more in-depth reading.

Let us start right away with the definition of a triangulated category.

Definition 1.32 *Let $\mathcal{D}$ be an additive category. The structure of a* triangulated category *on $\mathcal{D}$ is given by an additive equivalence*

$$T : \mathcal{D} \longrightarrow \mathcal{D},$$