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ABELIAN VARIETIES, THETA FUNCTIONS AND THE FOURIER TRANSFORM

ALEXANDER POLISHCHUK

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ABELIAN VARIETIES, THETA FUNCTIONS AND THE FOURIER TRANSFORM

This book is a modern introduction to the theory of abelian varieties and theta functions. Here the Fourier transform techniques play a central role, appearing in several different contexts. In transcendental theory, the usual Fourier transform plays a major role in the representation theory of the Heisenberg group, the main building block for the theory of theta functions. Also, the Fourier transform appears in the discussion of mirror symmetry for complex and symplectic tori, which are used to compute cohomology of holomorphic line bundles on complex tori. When developing the algebraic theory (in arbitrary characteristic), emphasis is placed on the importance of the Fourier–Mukai transform for coherent sheaves on abelian varieties. In particular, it is used in the computation of cohomology of line bundles, classification of vector bundles on elliptic curves, and proofs of the Riemann and Torelli theorems for Jacobians of algebraic curves.

Another subject discussed in the book is the construction of equivalences between derived categories of coherent sheaves on abelian varieties, which follows the same pattern as the construction of intertwining operators between different realizations of the unique irreducible representation of the Heisenberg group.

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Preface

In 1981, S. Mukai discovered a nontrivial algebro-geometric analogue of the Fourier transform in the context of abelian varieties, which is now called the *Fourier–Mukai transform* (see [7]). One of the main goals of this book is to present an introduction to the algebraic theory of abelian varieties in which this transform takes its proper place. In our opinion, the use of this transform gives a fresh point of view on this important theory. On the one hand, it allows one to give more conceptual proofs of the known theorems. On the other, the analogy with the usual Fourier analysis leads one to new directions in the study of abelian varieties. By coincidence, the standard Fourier transform usually appears in the proof of functional equation for theta functions; thus, it is relevant for analytic theory of complex abelian varieties. In references [6] and [9], this fact is developed into a deep relationship between theta functions and representation theory. In the first part of this book we present the basics of this theory and its connection with the geometry of complex abelian varieties. The algebraic theory of abelian varieties and of the Fourier–Mukai transform is developed in the second part. The third part is devoted to Jacobians of algebraic curves. These three parts are tied together by the theory of theta functions: They are introduced in Part I and then used in Parts II and III to illustrate abstract algebraic theorems. Part II depends also on Part I in a more informal way: An important role in the algebraic theory of abelian varieties is played by the theory of Heisenberg groups, which is an algebraic analogue of the corresponding theory in Part I.

Another motivation for our presentation of the theory of abelian varieties is the renewed interest on the Fourier–Mukai transform and its generalizations because of the recently discovered connections with the string theory, in particular, with mirror symmetry. Kontsevich's homological mirror conjecture predicts that for mirror dual pairs of Calabi–Yau manifolds there exists an equivalence of the derived category of coherent sheaves on one of these manifolds with certain category defined by using the symplectic structure on another manifold. This implies that the derived category of coherent sheaves on a Calabi–Yau manifold possessing a mirror dual has many autoequivalences. Such autoequivalences indeed often can be constructed, and the Fourier–Mukai transform is a typical example. On the other hand, it seems that the correspondence between coherent sheaves on a Calabi–Yau manifold and Lagrangian submanifolds in a mirror dual manifold predicted by the homological mirror conjecture should be given by an appropriate real analytic analogue of the Fourier–Mukai transform. At the end of Part I we sketch the simplest example of such a transform in the case of complex and symplectic tori.

We would like to stress that the important idea that influenced the structure of this book is the idea of *categorification* (see [1]). Roughly speaking, this is the process of finding category-theoretic analogues of set-theoretic concepts by replacing sets with categories, functions with functors, etc. The nontriviality of this (nonunique) procedure comes from the fact that axioms formulated as equalities should be replaced by isomorphisms, so one should in addition formulate compatibilities between these isomorphisms. Many concepts in the theory of abelian varieties turn out to be categorifications. For example, the category of line bundles and their isomorphisms on an abelian variety can be thought of as a categorification of the set of quadratic functions on an abelian group, the derived category of coherent sheaves on an abelian variety is a categorification of the space of functions on an abelian group, etc. Of course, the Fourier–Mukai transform is a categorification of the usual Fourier transform. In fact, the reader will notice that most of the structures discussed in Part II are categorifications of the structures from Part I. It is worth mentioning here that the idea of categorification was applied in some other areas of mathematics as well. The most spectacular example is the recent work of Khovanov [5] on the categorification of the Jones polynomial of knots.

Perhaps we need to emphasize that this book does not claim to provide an improvement of the existing accounts of the theory of abelian varieties and theta functions. Rather, its purpose is to enhance this classical theory with more recent ideas and to consider it in a slightly different perspective. For example, in our exposition of the algebraic theory of abelian varieties in Part II we did not try to include all the material contained in Mumford's book [8], which remains an unsurpassed textbook on the subject. Our choice of topics was influenced partly by their relevance for the theory of theta functions, which is a unifying theme for all three parts of the book, and partly by the idea of categorification. In Part I we were strongly influenced by the fundamental works of Lion and Vergne [6] and Mumford et al. [9]. However, our exposition is much more concise: We have chosen the bare minimum of ingredients that allow us to define theta series and to prove the functional equation for them. Our account of the theory of Jacobians in Part III is also far from complete, because our main idea was to stress the role of the Fourier–Mukai transform. Nevertheless, we believe that all main features of this theory are present in our exposition.

This book is based on the lectures given by the author at Harvard University in the fall of 1998 and Boston University in the spring of 2001. It is primarily intended for graduate students and postgraduate researchers interested in algebraic geometry. Prerequisites for Part I are basic complex and differential geometry as presented in chapter 0 of [3], basic Fourier analysis, and familiarity with main concepts of representation theory. Parts II and III are much more technical. For example, the definition of the Fourier–Mukai transform requires working with derived categories of coherent sheaves; to understand it, the reader should be familiar with the language of derived categories. References [2] and [10] can serve as a nice introduction to this language. We also assume the knowledge of algebraic geometry in the scope of the first four chapters of Hartshorne's book [4]. Occasionally, we use more complicated facts from algebraic geometry for which we give references. Some of these facts are collected in Appendix C. Each chapter ends with a collection of exercises. The results of some of these exercises are used in the main text.

Now let us describe the content of the book in more details. Chapters 1–7, which constitute Part I of the book, are devoted to the transcendental theory of abelian varieties. In Chapter 1 we classify holomorphic line bundles on complex tori. In Chapters 2–5 our main focus is the theory of theta functions. We show that they appear naturally as sections of holomorphic line bundles over complex tori. However, the most efficient tool for their study comes not from geometry but from representation theory. The relevant group is the Heisenberg group, which is a central extension of a vector space by $U(1)$, such that the commutator pairing is given by the exponent of a symplectic form. Theta functions arise when one compares different models for the unique irreducible representation of the Heisenberg group on which $U(1)$ acts in the standard way. The main result of this study is the functional equation for theta functions proved in Chapter 5. As another by-product of the study of the Heisenberg group representations, we prove in Appendix B some formulas for Gauss sums discovered by Van der Blij and Turaev.

In Chapters 6 and 7 we discuss mirror symmetry between symplectic tori and complex tori. The main idea is that for every symplectic torus equipped with a Lagrangian tori fibration, there is a natural complex structure on the dual tori fibration. Furthermore, there is a correspondence between Lagrangian submanifolds in a symplectic torus and holomorphic vector bundles on the mirror dual complex torus. The construction of this correspondence can be considered as a (toy) real version of the Fourier–Mukai transform. We apply these ideas to compute cohomology of holomorphic line bundles on complex tori.

Part II (Chapters 8–15) is devoted to algebraic theory of abelian varieties over an algebraically closed field of arbitrary characteristic. In Chapters 8–10 we study line bundles on abelian varieties and the construction of the dual abelian variety. Some of the material is a condensed review of the results from [8], chapter III, sections 10–15. However, the proof of the main theorem about duality of abelian varieties is postponed until Chapter 11, where we introduce the Fourier–Mukai transform. Another result proven in Chapter 11 is that line bundles on abelian varieties satisfying certain nondegeneracy condition have cohomology concentrated in one degree. The Fourier–Mukai transform is also applied to construct an action of a central extension of $\mathrm{SL}_2(\mathbb{Z})$ on the derived category of a principally polarized abelian variety. Then in Chapter 12 we develop an algebraic analogue of the representation theory of Heisenberg groups and apply it to the proof of Riemann's quartic theta identity. In Chapter 13 we revisit line bundles on abelian varieties and develop algebraic analogues of some structures associated to holomorphic line bundles on complex tori. Chapter 14 is devoted to the study of vector bundles on elliptical curves. The main idea is to combine the action of a central extension of $\mathrm{SL}_2(\mathbb{Z})$ on the derived category of sheaves on elliptic curve with the notion of stability of vector bundles. As a result, we recover Atiyah's classification of bundles on elliptical curves. In Chapter 15 we develop a categorification of the theory of representations of Heisenberg groups, in which the role of the usual Fourier transform is played by the Fourier–Mukai transform. The main result is a construction of equivalences between derived categories of coherent sheaves on abelian varieties, which "categorifies" the construction of intertwining operators between different models of the unique representation of the Heisenberg group given in Chapter 4.

In Part III (Chapters 16–22) we present some topics related to Jacobians of algebraic curves. Chapter 16 is devoted to the construction of the Jacobian of a curve by gluing open pieces of its gth symmetric power, where g is the genus of the curve. We also present some basic results on symmetric powers of curves with proofs that work in arbitrary characteristic. In Chapter 17 we define the principal polarization on the Jacobian and give a modern treatment to some classical topics related to the geometry of the embedding of a curve in its Jacobian. In particular, we prove Riemann's theorem describing intersections of the curve with theta divisors in the Jacobian. Chapter 18 is devoted to the proof of Fay's trisecant identity, which is a special identity satisfied by theta functions on the Jacobian. The proof is a combination of the theory developed in Chapter 17 with the residue theorem for rational differentials on a curve. In Chapter 19 we present a more detailed study of the symmetric powers of a curve. The main results are the calculation of the Picard groups and the vanishing theorem for cohomology of some natural vector bundles. We also study Chern classes of the vector bundles over the Jacobian whose projectivizations are isomorphic to symmetric powers of the curve. Chapter 20 is devoted to the varieties of special divisors. Its main results are estimates on dimensions of these varieties and an explicit description of tangent cones to their singular points. In Chapter 21 we prove the Torelli theorem stating that a curve can be recovered from its Jacobian and the theta divisor in it. The idea of the proof is to use the fact that the Fourier–Mukai transform exchanges some coherent sheaves supported on the curve embedded into its Jacobian with coherent sheaves supported on the theta divisor. Finally, in Chapter 22 we discuss Deligne's symbol for a pair of line bundles on a relative curve and its relation to the principal polarization of the Jacobian. We also take a look at the strange duality conjecture, which involves generalization of theta functions to moduli space of vector bundles on curves. The main result is a reformulation of this conjecture in a symmetric way by using the Fourier–Mukai transform.

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Part I

Analytic Theory

1

Line Bundles on Complex Tori

In this chapter we study holomorphic line bundles on complex tori, i.e., quotients of complex vector spaces by integral lattices. The main result is an explicit description of the group of isomorphism classes of holomorphic line bundles on a complex torus T. The topological type of a complex line bundle L on T is determined by its first Chern class $c_1(L) \in H^2(T, \mathbb{Z})$. This cohomology class can be interpreted as a skew-symmetric bilinear form $E : \Gamma \times \Gamma \to \mathbb{Z}$, where $\Gamma = H_1(T, \mathbb{Z})$ is the lattice corresponding to T. The existence of a holomorphic structure on L is equivalent to the compatibility of E with the complex structure on $\Gamma \otimes \mathbb{R}$ by which we mean the identity $E(iv, iv') = E(v, v')$. On the other hand, the group of isomorphism classes of topologically trivial holomorphic line bundles on T can be easily identified with the dual torus $T^\vee = \mathrm{Hom}(\Gamma, U(1))$. Now the set of isomorphism classes of holomorphic line bundles on T with the fixed first Chern class E is a $T^\vee$-torsor[1]. It can be identified with the $T^\vee$-torsor of quadratic maps $\alpha : \Gamma \to U(1)$ whose associated bilinear map $\Gamma \times \Gamma \to U(1)$ is equal to $\exp(\pi i E)$. These results provide a crucial link between the theory of theta functions and geometry that will play an important role throughout the first part of this book.

The holomorphic line bundle on T corresponding to a skew-symmetric form E and a quadratic map α as above, is constructed explicitly by equipping the trivial line bundle over a complex vector space with an action of an integral lattice. We show that as a result, every holomorphic line bundle on T has a canonical Hermitian metric and a Hermitian connection. We also show that the dual torus, $T^\vee$, has a natural complex structure and the universal family $\mathcal{P}$ of line bundles on T parametrized by $T^\vee$ (called the *Poincaré bundle*) has a natural holomorphic structure that we describe. In Chapter 9 we will study a purely algebraic version of this duality for abelian varieties.

[1] Following Grothendieck, we will use the term *G-torsor* when referring to a principal homogeneous space for a group G.

1.1. Cohomology of the Structure Sheaf

Let V be a finite-dimensional complex vector space, Γ be a lattice in V (i.e., Γ is a finitely generated $\mathbb{Z}$-submodule of V such that the natural map $\mathbb{R} \otimes_{\mathbb{Z}} \Gamma \to V$ is an isomorphism).

Definition. The complex manifold $T = V/\Gamma$ is called a *complex torus*.

As a topological space T is just a product of circles, so the cohomology ring $H^*(T, \mathbb{Z}) = \oplus_r H^r(T, \mathbb{Z})$ (resp., $H^*(T, \mathbb{R})$) can be identified naturally with the exterior algebra $\bigwedge^* H^1(T, \mathbb{Z})$ (resp., $\bigwedge^* H^1(T, \mathbb{R})$). Furthermore, we have a natural isomorphism $\Gamma \widetilde{\to} H_1(T, \mathbb{Z})$ sending $\gamma \in \Gamma$ to the cycle $\mathbb{R}/\mathbb{Z} \to T : t \mapsto t\gamma$. Therefore, we get canonical isomorphisms $H^*(T, \mathbb{Z}) \simeq \bigwedge^* \Gamma^\vee$ and $H^*(T, \mathbb{R}) \simeq \bigwedge^* \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, where $\Gamma^\vee = \mathrm{Hom}(\Gamma, \mathbb{Z})$ is the lattice dual to Γ.

Recall that one has the direct sum decomposition

$$V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \overline{V},$$

where V is identified with the subset of $V \otimes_{\mathbb{R}} \mathbb{C}$ consisting of vectors of the form $v \otimes 1 - iv \otimes i$, $\overline{V}$ is the complex conjugate subspace consisting of vectors $v \otimes 1 + iv \otimes i$. We also have the corresponding decomposition

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^\vee \oplus \overline{V}^\vee,$$

where $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the dual complex vector space to V, $\overline{V}^\vee$ is the space of $\mathbb{C}$-antilinear functionals on V. Since T is a Lie group, the tangent bundle to T is trivial and the above decomposition is compatible with the decomposition of the bundle of complex valued 1-forms on T according to types (1, 0) and (0, 1). Hence, we have canonical isomorphisms

$$\mathcal{E}^{p,q} \simeq \bigwedge^p V^\vee \otimes_{\mathbb{C}} \bigwedge^q \overline{V}^\vee \otimes_{\mathbb{C}} \mathcal{E}^{0,0},$$

where $\mathcal{E}^{p,q}$ is the sheaf of smooth (p, q)-forms on T.

The first basic result about T as a complex manifold is the calculation of cohomology of the structure sheaf $\mathcal{O}$, i.e., the sheaf of holomorphic functions.

Proposition 1.1. *One has a canonical isomorphism* $H^r(T, \mathcal{O}) \simeq \bigwedge^r \overline{V}^\vee$.

Proof. To calculate cohomology of $\mathcal{O}$ one can use the Dolbeault resolution:

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1} \xrightarrow{\overline{\partial}} \mathcal{E}^{0,2} \to \cdots.$$

We can consider elements of $\bigwedge^p \overline{V}^\vee$ as translation-invariant $(0, p)$-forms on T. Note that translation-invariant forms are automatically closed. We claim that this gives an embedding

$$i : \bigwedge^p \overline{V}^\vee \hookrightarrow H^p(T, \mathcal{O}).$$

Indeed, let $\int : \mathcal{E}^{0,0} \to \mathbb{C}$ be the integration map (with respect to some translation-invariant volume form on T) normalized by the condition $\int 1 = 1$. Then we can extend $\int$ to the map $\int : \mathcal{E}^{0,p} \to \bigwedge^p \overline{V}^\vee$. It is easy to see that $\int \circ \overline{\partial} = 0$, so $\int$ induces the map on cohomology

$$\int : H^p(T, \mathcal{O}) \to \bigwedge^p \overline{V}^\vee$$

such that $\int \circ i = \text{id}$. Hence, i is an embedding. Let Ω^q be the sheaf of holomorphic q-forms on T. Since $\Omega^q \simeq \bigwedge^q V^\vee \otimes \mathcal{O}$, there is an induced embedding

$$i : \oplus_{p,q} \bigwedge^q V^\vee \otimes \bigwedge^p \overline{V}^\vee \to \oplus_{p,q} H^p(T, \Omega^q).$$

Notice that the source of this embedding can be identified with $H^*(T, \mathbb{C}) \simeq \bigwedge^*(V^\vee \oplus \overline{V}^\vee)$. Recall that for every Kähler complex compact manifold X one has Hodge decomposition $H^*(X, \mathbb{C}) \simeq \oplus_{p,q} H^p(X, \Omega^q)$ (e.g., [52], Chapter 0, Section 7). Since any translation-invariant Hermitian metric on T is Kähler, it follows that $\dim H^*(T, \mathbb{C}) = \dim \oplus_{p,q} H^p(T, \Omega^q)$. Therefore, the embedding i is an isomorphism. □

1.2. Appell–Humbert Theorem

It is well known that all holomorphic line bundles on $\mathbb{C}^n$ are trivial. Indeed, from the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \stackrel{\exp}{\to} \mathcal{O}^* \to 0 \tag{1.2.1}$$

we see that it suffices to prove triviality of $H^1(\mathbb{C}^n, \mathcal{O})$. But $H^{>0}(\mathbb{C}^n, \mathcal{O}) = 0$ by Poincaré $\overline{\partial}$-lemma ([52], Chapter 0, Section 2.)

For every complex manifold X we denote by $\text{Pic}(X)$ the Picard group of X, i.e., the group of isomorphism classes of holomorphic line bundles on X. Triviality of $\text{Pic}(\mathbb{C}^n)$ leads to the following computation of $\text{Pic}(T)$ in terms of group cohomology of the lattice Γ.

Proposition 1.2. *Every holomorphic line bundle L on T is a quotient of the trivial bundle over V by the action of Γ of the form $\gamma(z, v) = (e_\gamma(v)z, v + \gamma)$,*