

Linear Programming, 1: Introduction

*George B. Dantzig
Mukund N. Thapa*

Springer

George B. Dantzig Mukund N. Thapa

Linear Programming

1: Introduction

With 87 Illustrations



Springer

Springer Series in Operations Research

Editor:
Peter Glynn

Springer

New York
Berlin
Heidelberg
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Santa Clara
Singapore
Tokyo

Springer Series in Operations Research

Altiok: Performance Analysis of Manufacturing Systems

Dantzig and Thapa: Linear Programming 1: Introduction

Drezner (Editor): Facility Location: A Survey of Applications
and Methods

Fishman: Monte Carlo: Concepts, Algorithms, and Applications

Olson: Decision Aids for Selection Problems

Yao (Editor): Stochastic Modeling and Analysis of Manufacturing
Systems

George B. Dantzig
Professor of Operations Research
and Computer Science
Department of Operations Research
Stanford University
Stanford, CA 94305
USA

Mukund N. Thapa
President
Stanford Business Software, Inc.
Suite 304
2680 Bayshore Parkway
Mountain View, CA 94043
and
Consulting Professor of Operations
Research
Stanford University
Stanford, CA 94305
USA

Series Editor:
Peter Glynn
Department of Operations Research
Stanford University
Stanford, CA 94305
USA

Library of Congress Cataloging-in-Publication Data
Dantzig, George Bernard, 1914-

Linear programming 1 : introduction / George B. Dantzig
& Mukund N. Thapa.

p. cm. — (Springer series in operations research)

Includes bibliographical references and index.

ISBN 0-387-94833-3 (hardcover : alk. paper)

1. Linear programming. I. Thapa, Mukund Narain-Dhami. II. Title.
III. Series.

T57.74.D365 1997

619.7'2—dc20

96-36411

© 1997 George B. Dantzig and Mukund N. Thapa

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

ABOUT THE AUTHORS

George B. Dantzig received the National Medal of Science from the President of the United States “for inventing Linear Programming and for discovering the Simplex Algorithm that led to wide-scale scientific and technical applications to important problems in logistics, scheduling, and network optimization, and to the use of computers in making efficient use of the mathematical theory.” He is world famous for his twin discoveries; linear programming and the Simplex Algorithm, which together have enabled mankind for the first time to structure and solve extremely complex optimal allocation and resource problems. Among his other discoveries are the Decomposition Principle (with Philip Wolfe) which makes it possible to decompose and solve extremely large linear programs having special structures, and applications of these techniques with sampling to solving practical problems subject to uncertainty.

Since its discovery in 1947, the field of linear programming, together with its extensions (mathematical programming), has grown by leaps and bounds and is today the most widely used tool in industry for planning and scheduling.

George Dantzig received his master’s from Michigan and his doctorate in mathematics from Berkeley in 1946. He worked for the U.S. Bureau of Labor Statistics, served as chief of the Combat Analysts Branch for USAF Headquarters during World War II, research mathematician for RAND Corporation, and professor and head of the Operations Research Center at the University of California, Berkeley. He is currently professor of operations research and computer science at Stanford University. He served as director of the System Optimization Laboratory and the PILOT Energy-Economic Model Project. Professor Dantzig’s seminal work has laid the foundation for the field of systems engineering, which is widely used in network design and component design in computer, mechanical, and electrical engineering. His work inspired the formation of the Mathematical Programming Society, a major section of the Society of Industrial and Applied Mathematics, and numerous professional and academic bodies. Generations of Professor Dantzig’s students have become leaders in industry and academia.

He is a member of the prestigious National Academy of Science, the American Academy of Arts and Sciences, and the National Academy of Engineering.

Mukund N. Thapa is the president of Stanford Business Software, Inc., as well as a consulting professor of operations research at Stanford University. He received a bachelor of technology degree in metallurgical engineering from the Indian Institute of Technology, Bombay, and M.S. and Ph.D. degrees in operations research from Stanford University. His Ph.D. thesis was concerned with developing specialized algorithms for solving large-scale unconstrained nonlinear minimization problems. By profession he is a software developer who produces commercial software products as well as commercial-quality custom software. Since 1978, Dr. Thapa has been applying the theory of operations research, statistics, and computer science to develop efficient, practical, and usable solutions to a variety of problems.

At Stanford Business Software, Dr. Thapa, ensures that the company produces high-quality turnkey software for clients. His expert knowledge of user-friendly interfaces, data bases, computer science, and modular software design plays an important role in making the software practical and robust. His speciality is the application of numerical analysis methodology to solve mathematical optimization problems. He is also an experienced modeler who is often asked by clients to consult, prepare analyses, and to write position papers. At the Department of Operations Research, Dr. Thapa teaches graduate-level courses in mathematical programming computation and numerical methods of linear programming.

TO

Tobias and Anja Dantzig, my parents, *in memoriam*,
Anne S. Dantzig, my wife, and to
the great pioneers who made this field possible:
Wassily Leontief, Tjalling Koopmans, John von Neumann,
Albert Tucker, William Orchard-Hays, Martin Beale.

— George B. Dantzig

Devi Thapa and Narain S. Thapa, my parents,
and Radhika H. Thapa, my wife.

— Mukund N. Thapa

This page intentionally left blank

Contents

FOREWORD	xxi
PREFACE	xxxiii
DEFINITION OF SYMBOLS	xxxvii
1 THE LINEAR PROGRAMMING PROBLEM	1
1.1 SOME SIMPLE EXAMPLES	2
1.2 MATHEMATICAL STATEMENT	7
1.3 FORMULATING LINEAR PROGRAMS	8
1.3.1 The Column (Recipe/Activity) Approach	9
1.3.2 The Row (Material Balance) Approach	11
1.4 EXAMPLES OF MODEL FORMULATION	12
1.4.1 Product Mix Problem (Column Approach)	12
1.4.2 Product Mix Problem (Row Approach)	15
1.4.3 A Simple Warehouse Problem	16
1.4.4 On-the-Job Training	18
1.5 BOUNDS	21
1.6 AXIOMS	22
1.7 NOTES & SELECTED BIBLIOGRAPHY	23
1.8 PROBLEMS	25
2 SOLVING SIMPLE LINEAR PROGRAMS	35
2.1 TWO-VARIABLE PROBLEM	35
2.2 TWO-EQUATION PROBLEM	37
2.2.1 Graphical Solution	38
2.2.2 The Dual Linear Program	41
2.3 FOURIER-MOTZKIN ELIMINATION	43
2.3.1 Illustration of the FME Process	44
2.3.2 The Fourier-Motzkin Elimination Algorithm	46
2.3.3 Fourier-Motzkin Elimination Theory	47
2.4 INFEASIBILITY THEOREM	52
2.5 NOTES & SELECTED BIBLIOGRAPHY	53

2.6	PROBLEMS	54
3	THE SIMPLEX METHOD	63
3.1	GRAPHICAL ILLUSTRATION	64
3.2	THE SIMPLEX ALGORITHM	64
3.2.1	Canonical Form and Basic Variables	64
3.2.2	Improving a Nonoptimal Basic Feasible Solution	68
3.2.3	The Simplex Algorithm	71
3.2.4	Theory Behind the Simplex Algorithm	73
3.3	SIMPLEX METHOD	76
3.3.1	The Method	77
3.3.2	Phase I/Phase II Algorithm	78
3.3.3	Theory Behind Phase I	81
3.4	BOUNDED VARIABLES	83
3.5	REVISED SIMPLEX METHOD	89
3.5.1	Motivation	89
3.5.2	Revised Simplex Method Illustrated	92
3.5.3	Revised Simplex Algorithm	93
3.5.4	Computational Remarks	96
3.6	NOTES & SELECTED BIBLIOGRAPHY	97
3.7	PROBLEMS	98
4	INTERIOR-POINT METHODS	113
4.1	BASIC CONCEPTS	115
4.2	PRIMAL AFFINE / DIKIN'S METHOD	118
4.3	INITIAL SOLUTION	121
4.4	NOTES & SELECTED BIBLIOGRAPHY	122
4.5	PROBLEMS	124
5	DUALITY	129
5.1	DUAL AND PRIMAL PROBLEMS	129
5.1.1	Von Neumann Symmetric Form	129
5.1.2	Tucker Diagram	130
5.1.3	Duals of Mixed Systems	130
5.1.4	The Dual of the Standard Form	132
5.1.5	Primal-Dual Feasible-Infeasible Cases	133
5.2	DUALITY THEOREMS	134
5.3	COMPLEMENTARY SLACKNESS	135
5.4	OBTAINING A DUAL SOLUTION	136
5.5	NOTES & SELECTED BIBLIOGRAPHY	138
5.6	PROBLEMS	139

6	EQUIVALENT FORMULATIONS	145
6.1	RESTRICTED VARIABLES	145
6.2	UNRESTRICTED (FREE) VARIABLES	146
6.3	ABSOLUTE VALUES	147
6.4	GOAL PROGRAMMING	150
6.5	MINIMIZING THE MAXIMUM OF LINEAR FUNCTIONS	152
6.6	CURVE FITTING	154
6.7	PIECEWISE LINEAR APPROXIMATIONS	157
6.7.1	Convex/Concave Functions	157
6.7.2	Piecewise Continuous Linear Functions	159
6.7.3	Separable Piecewise Continuous Linear Functions	160
6.8	NOTES & SELECTED BIBLIOGRAPHY	162
6.9	PROBLEMS	162
7	PRICE MECHANISM AND SENSITIVITY ANALYSIS	171
7.1	THE PRICE MECHANISM OF THE SIMPLEX METHOD	172
7.1.1	Marginal Values or Shadow Prices	173
7.1.2	Economic Interpretation of the Simplex Method	174
7.1.3	The Manager of a Machine Tool Plant	175
7.1.4	The Ambitious Industrialist	181
7.1.5	Sign Convention on Prices	183
7.2	INTRODUCING A NEW VARIABLE	184
7.3	INTRODUCING A NEW CONSTRAINT	186
7.4	COST RANGING	188
7.5	CHANGES IN THE RIGHT-HAND SIDE	190
7.6	CHANGES IN THE COEFFICIENT MATRIX	192
7.7	THE SUBSTITUTION EFFECT OF NONBASIC ACTIVITIES ON BASIC ACTIVITIES	198
7.8	NOTES AND SELECTED BIBLIOGRAPHY	199
7.9	PROBLEMS	199
8	TRANSPORTATION AND ASSIGNMENT PROBLEM	205
8.1	THE CLASSICAL TRANSPORTATION PROBLEM	205
8.1.1	Mathematical Statement	206
8.1.2	Properties of the System	206
8.2	STANDARD TRANSPORTATION ARRAY	212
8.3	FINDING AN INITIAL SOLUTION	214
8.3.1	Triangularity Rule	214
8.3.2	The Least Remaining Cost Rule	217
8.3.3	Vogel's Approximation Method	217
8.3.4	Russel's Approximation Method	218
8.3.5	Cost Preprocessing	219
8.4	FAST SIMPLEX ALGORITHM FOR THE TRANSPORTATION PROBLEM	222
8.4.1	Simplex Multipliers, Optimality, and the Dual	222

8.4.2	Finding a Better Basic Solution	224
8.4.3	Illustration of the Solution Process	225
8.5	THE ASSIGNMENT PROBLEM	229
8.6	EXCESS AND SHORTAGE	233
8.6.1	Mathematical Statement	234
8.6.2	Properties of the System	236
8.6.3	Conversion to the Classical Form	236
8.6.4	Simplex Multipliers and Reduced Costs	238
8.7	PRE-FIXED VALUES AND INADMISSIBLE SQUARES	239
8.8	THE CAPACITATED TRANSPORTATION PROBLEM	240
8.9	NOTES & SELECTED BIBLIOGRAPHY	244
8.10	PROBLEMS	245
9	NETWORK FLOW THEORY	253
9.1	TERMINOLOGY	253
9.2	FLOWS AND ARC-CAPACITIES	258
9.3	AUGMENTING PATH ALGORITHM FOR MAXIMAL FLOW	262
9.4	CUTS IN A NETWORK	275
9.5	SHORTEST ROUTE	277
9.6	MINIMAL SPANNING TREE	282
9.7	MINIMUM COST-FLOW PROBLEM	286
9.8	THE NETWORK SIMPLEX METHOD	288
9.9	THE BOUNDED VARIABLE PROBLEM	299
9.10	NOTES & SELECTED BIBLIOGRAPHY	301
9.11	PROBLEMS	304
A	LINEAR ALGEBRA	315
A.1	SCALARS, VECTORS, AND MATRICES	315
A.2	ARITHMETIC OPERATIONS WITH VECTORS AND MATRICES	317
A.3	LINEAR INDEPENDENCE	320
A.4	ORTHOGONALITY	321
A.5	NORMS	321
A.6	VECTOR SPACES	324
A.7	RANK OF A MATRIX	326
A.8	MATRICES WITH SPECIAL STRUCTURE	326
A.9	INVERSE OF A MATRIX	329
A.10	INVERSES OF SPECIAL MATRICES	330
A.11	DETERMINANTS	331
A.12	EIGENVALUES	333
A.13	POSITIVE-DEFINITENESS	336
A.14	NOTES & SELECTED BIBLIOGRAPHY	337
A.15	PROBLEMS	337

B	LINEAR EQUATIONS	341
B.1	SOLUTION SETS	341
B.2	SYSTEMS OF EQUATIONS WITH THE SAME SOLUTION SETS	343
B.3	HOW SYSTEMS ARE SOLVED	345
B.4	ELEMENTARY OPERATIONS	346
B.5	CANONICAL FORMS, PIVOTING, AND SOLUTIONS	349
B.6	PIVOT THEORY	354
B.7	NOTES & SELECTED BIBLIOGRAPHY	357
B.8	PROBLEMS	357
	REFERENCES	361

This page intentionally left blank

List of Figures

1-1	Manufacturing Activity 1	13
1-2	Slack Activity 5	14
1-3	Input-Output Characteristics for the Warehouse Problem	17
1-4	Activities for the On-the-Job Training Problem	19
2-1	Graphical Solution of a Two-Variable LP	36
2-2	Graphical Solution of the Product Mix Problem	40
2-3	Optimality Check—The Product Mix Problem	42
3-1	Graphical Solution of a Two-Variable LP	65
4-1	Comparison of a Move from a Point \hat{x}^t Near the Center Versus a Point \bar{x}^t Near the Boundary.	115
5-1	Illustration of the Duality Gap	135
6-1	Absolute Value Function	147
6-2	Examples of Convex and General Functions	158
6-3	Piecewise Continuous Linear Functions	159
6-4	Purchase Price of Raw Milk in Region A	166
6-5	Separation of Region A Milk	167
6-6	Purchase Price of Raw Milk in Region B	167
6-7	Separation of Region B Milk	168
8-1	Network Representation of the Transportation Problem	207
8-2	Illustration of the Basis Triangularity Theorem	211
8-3	Example of a Standard Transportation Array	213
8-4	Transportation Array Example	214
8-5	Buy from the Cheapest Source	215
8-6	Illustration of the Triangularity Rule	216
8-7	Illustration of the Northwest Corner Rule	217
8-8	Illustration of the Least Remaining Cost Rule	218
8-9	Illustration of Vogel's Approximation Method	219
8-10	Illustration of Russel's Approximation Method	220
8-11	Cost Preprocessing	221