Huaxin Lin

An Introduction to the Classification of Amenable C*-Algebras

World Scientific

An Introduction to the Classification of

Amenable C*-Algebras

This page is intentionally left blank

An Introduction to the Classification of
Amenable C*-Algebras

Huaxin Lin
University of Oregon, USA

World Scientific
New Jersey • London • Singapore • Hong Kong • Bangalore

Published by

World Scientific Publishing Co. Pte. Ltd.
P O Box 128, Farrer Road, Singapore 912805
USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

AN INTRODUCTION TO THE CLASSIFICATION OF AMENABLE C*-ALGEBRAS

ISBN 981-02-4680-3

Printed in Singapore by Mainland Press

To whom I love

This page is intentionally left blank

Preface

The theory and applications of C^*-algebras are related to such diverse fields as operator theory, group representations, topology, quantum mechanics, non-commutative geometry and dynamical systems. In light of the Gelfand transformation, the theory of C^*-algebras is also regarded as non-commutative topology. Despite the great influence of this subject to other fields, the understanding of C^*-algebras itself was very limited. About a decade ago, George A. Elliott initiated the program of classification of C^*-algebras (up to isomorphism) by their K-theoretical data. It started with the classification of $A\mathbb{T}$ -algebras with real rank zero. Since then, great efforts have been made to classify amenable C^*-algebras, a class of C^*-algebras that appears most naturally. Large classes of simple amenable C^*-algebras were discovered to be classifiable. With these rapid development, the theory of C^*-algebras becomes increasely important to many other fields. For example, the applications of these results to dynamical systems have been well established.

The purpose of this book is to introduce some of the recent developments of the theory of classification of amenable C^*-algebras to a broad range of readers including non-experts and graduate students. It is an ambitious plan. However, the material presented here has been limited by the author's knowledge as well as the page limitation of this volume. For example, the aspects of classification of purely infinite simple C^*-algebras which is quite complete are not mentioned in this volume. The author's effort was concentrated to finite C^*-algebras. Even in this case, only simple C^*-algebras with tracial topological rank zero are treated in detail.

The first three chapters contain the basics of the theory of C^*-algebras which are particularly important to the theory of the classification of amenable C^*-algebras. References to these three chapters include (but not limited to) [147], [173], [143] and [48]. Chapter 4 offers the classification of the so-called $A\mathbb{T}$-algebras of real rank zero. The results in Chapter 6 cover the results in Chapter 4, however, the proofs given in Chapter 4 are much more elementary. It is the author's intention to present the classification of simple $A\mathbb{T}$-algebras of real rank zero with limited tools so non-experts and graduate students may be able to read it without advanced knowledge of C^*-algebras and K-theory. The first four chapters and first 6 sections of Chapter 5 are self-contained. This part could serve as a text book for a graduate course on C^*-algebras. Indeed the author used it for a graduate course in University of Oregon and a lecture series in East China Normal University. The last two chapters contain more advanced topics. In particular, they contain the classification theorem for simple AH-algebras with real rank zero, the work of Elliott and Gong. To achieve these goals in such a limited volume, the author was often forced to give some new proofs (to avoid introducing too much new concepts and materials). Starting Chapter 4, at the end of each chapter, brief remarks are inserted. The intention is to give the reader some rough idea of the development related to the material presented there. They are bound to contain errors. The author asks forgiveness from those experts whose works have not been mentioned.

The majority of this book was written when the author was visiting East China Normal University during the summer of 2000, the Mathematical Sciences Research Institute at Berkeley during the fall of 2000 and University of California at Santa Barbara in spring of 2001. It is his pleasure to express his gratitude for all hospitalities he received at these institutes. The exciting environment at the MSRI has left him with a great memories far beyond this book. Even though it is difficult for the author to express his exact appreciation to the people whom he met at MSRI, now is perhaps the only opportunity to record his sincere appreciation. The author thanks Professor N. C. Phillips and Shuang Zhang for their comments and suggestions. He would also like to take this opportunity to thank following persons who were exposed to earlier drafts and made corrections, comments and suggestions: Warren Akers, Shanwen Hu, Bobby Ilapogu, Benjamin Itza-Ortiz, Junping Liu, Shudong Liu, Nancy Livingston, Lisa Oberbroeckling, Michael Raney and Tadg Woods.

Contents

朝辞白帝彩云间

— 759AD — 李白 (701 - 762)

Chapter 1

The Basics of C^*-algebras

1.1 Banach algebras

Definition 1.1.1 A *normed algebra* is a complex algebra A which is a normed space, and the norm satisfies

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

If A (with this norm) is complete, then A is called a *Banach algebra.*

Every closed subalgebra of a Banach algebra is itself a Banach algebra.

Example 1.1.2 Let $\mathbb{C}$ be the complex field. Then $\mathbb{C}$ is a Banach algebra. Let X be a compact Hausdorff space and $C(X)$ the set of continuous functions on X. $C(X)$ is a complex algebra with pointwise operations. With $\|f\| = \sup_{x\in X} |f(x)|$, $C(X)$ is a Banach algebra.

Example 1.1.3 Let M_n be the algebra of $n \times n$ complex matrices. By identifying M_n with $B(\mathbb{C}^n)$, the set of all (bounded) linear maps from the n-dimensional Hilbert space $\mathbb{C}^n$ to $\mathbb{C}^n$, with operator norm, i.e, $\|x\| = \sup_{\xi\in\mathbb{C}^n, \|\xi\|\leq 1} \|x(\xi)\|$, we see that M_n is a Banach algebra.

Example 1.1.4 The set of continuous functions $A(\mathbf{D})$ on the closed unit disk $\mathbf{D}$ in the plane which are analytic on the interior is a closed subalgebra of $C(\mathbf{D})$. Therefore, $A(\mathbf{D})$ is a Banach algebra.

Example 1.1.5 Let X be a Banach space and $B(X)$ be the set of all bounded linear operators on X. If $T, L \in B(X)$, define $TL = T \circ L$. Then $B(X)$ is a complex algebra. With operator norm, $B(X)$ is a Banach algebra.

A *commutative* Banach algebra is a Banach algebra A with the property that $ab = ba$ for all $a, b \in A$ Examples 1.1.2 and 1.1.4 are of commutative Banach algebras while Example 1.1.3 1.1.5 are not commutative.

Definition 1.1.6 In a unital algebra, an element $a \in A$ is called *invertible* if there is an element $b \in A$ such that $ab = ba = 1$. In this case b is unique and written a^{-1}. The set

$$GL(A) = \{a \in A : a \text{ is invertible}\}$$

is a group under multiplication.

We define the *spectrum* of an element a to be the set

$$\mathrm{sp}(a) = sp_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin GL(A)\}.$$

Whenever there is no confusion, we will write $\lambda 1$ simply as λ.

The complement of the spectrum is called the *resolvent* and $R(\lambda) = (\lambda - a)^{-1}$ is the *resolvent function.*

Example 1.1.7 Let $A = C(X)$ be as in 1.1.2. Then $\mathrm{sp}(f) = f(X)$ for all $f \in A$. In other words, the spectrum of f is the range of f.

Let $A = M_n$. If $a = (a_{ij}) \in A$, then the reader can check that $\mathrm{sp}(a)$ is the set of eigenvalues of the matrix a.

Proposition 1.1.8 *For any a and b in A,*

$$\mathrm{sp}(ab) \setminus \{0\} = \mathrm{sp}(ba) \setminus \{0\}.$$

Proof. If $\lambda \notin \mathrm{sp}(ab)$ and $\lambda \neq 0$, then there is $c \in A$ such that

$$c(\lambda - ab) = (\lambda - ab)c = 1.$$

Thus $c(ab) = \lambda c - 1 = (ab)c$. So we compute that

$$\begin{aligned}(1 + bca)(\lambda - ba) &= \lambda - ba + \lambda bca - bcaba \\ &= \lambda - ba + b(\lambda - ab)ca = \lambda - ba + ba = \lambda,\end{aligned}$$

which shows that $\lambda^{-1}(1 + bca)$ is the inverse of $\lambda - ba$. Hence $\lambda \notin \mathrm{sp}(ba)$ and $\mathrm{sp}(ba) \setminus \{0\} \subset \mathrm{sp}(ab) \setminus \{0\}$. □

Definition 1.1.9 A Banach algebra A is said to be *unital* if it admits a unit 1 and $\|1\| = 1$. Banach algebras in 1.1.2, 1.1.3 and 1.1.4 are unital.

Lemma 1.1.10 *Let A be a unital Banach algebra and a be an element of A such that $\|1-a\| < 1$. Then $a \in GL(A)$ and*

$$a^{-1} = \sum_{n=0}^{\infty} (1-a)^n.$$

Moreover, $\|a^{-1}\| \leq \frac{1}{1-\|1-a\|}$ *and* $\|1-a^{-1}\| \leq \frac{\|1-a\|}{1-\|1-a\|}$.

Proof. Since

$$\sum_{n=0}^{\infty} \|(1-a)^n\| \leq \sum_{n=0}^{\infty} \|1-a\|^n = \frac{1}{(1-\|1-a\|)} < \infty,$$

the series $\sum_{n=0}^{\infty}(1-a)^n$ is convergent. Let b be its limit in A. Then $\|b\| \leq \frac{1}{(1-\|1-a\|)}$ and

$$\|1-b\| \leq \sum_{n=1}^{\infty} \|1-a\|^n = \frac{\|1-a\|}{1-\|1-a\|}.$$

One verifies

$$a(\sum_{n=0}^{k}(1-a)^n) = (1-(1-a))(\sum_{n=0}^{k}(1-a)^n) = 1-(1-a)^{k+1}$$

and that it converges to $ab = ba = 1$ as $k \to \infty$. Hence b is the inverse of a.

□

Definition 1.1.11 A function f from an open subset $\Omega \subset \mathbb{C}$ to a Banach algebra is said to be *analytic*, if for any $\lambda_0 \in \Omega$ there is an open neighborhood $O(\lambda_0)$ such that $f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda-\lambda_0)^n$ converges for every $\lambda \in O(\lambda_0)$. To include the case that $\lambda_0 = \infty$, we say f is analytic at infinity if $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}$ for all λ in a neighborhood of infinity.

Theorem 1.1.12 *In any unital Banach algebra A, the spectrum of each $a \in A$ is a non-empty compact subset, and the resolvent function is analytic on $\mathbb{C} \setminus \mathrm{sp}(a)$.*

Proof. If $|\lambda| > \|a\|$, then $\|\lambda^{-n}a^n\| \leq (\frac{\|a\|}{\lambda})^n$. So the series

$$\sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$$

converges (in norm). Similarly to 1.1.10,

$$(\lambda - a)\sum_{n=0}^{k} \frac{a^n}{\lambda^{n+1}} = 1 - (\frac{a^{k+1}}{\lambda^{k+1}})$$

which converges to 1. This shows that $\sup_{\lambda \in \mathrm{sp}(a)} |\lambda| \le \|a\|$ and $R(\lambda)$ is analytic in $\{\lambda : |\lambda| > \|a\|\}$. Moreover,

$$\lim_{|\lambda| \to \infty} \|R(\lambda)\| \le \lim_{|\lambda| \to \infty} \frac{|\lambda|^{-1}}{1 - (\frac{\|a\|}{\lambda})} = 0. \qquad \text{(e 1.1)}$$

Similarly, if $\lambda_0 - a$ is invertible and $|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 - a)^{-1}\|}$, then

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n ((\lambda_0 - a)^{-1})^{n+1}.$$

This also shows that the resolvent is open. Since $\mathrm{sp}(a)$ has been shown to be bounded, $\mathrm{sp}(a)$ is compact. We have also shown that the resolvent function is analytic on the complement of the spectrum.

In particular, $f(R(\lambda))$ is a (scalar) analytic function for every bounded linear functional $f \in A^*$. If $\mathrm{sp}(a)$ were empty, then $f(R(\lambda))$ would be an entire function for every $f \in A^*$. However, (e 1.1) shows that $f(R(\lambda))$ is bounded on the plane. Thus Liouville's theorem implies that $f(R(\lambda))$ is a constant. But (e 1.1) also implies that $f(R(\lambda)) = 0$. So, by the Hahn-Banach theorem, $R(\lambda) = 0$. This is a contradiction. Hence $\mathrm{sp}(a)$ is not empty. □

Corollary 1.1.13 *The only simple commutative unital Banach algebra is* $\mathbb{C}$.

Proof. Suppose that A is a unital commutative Banach algebra and $a \in A$ is not a scalar. Let $\lambda \in \mathrm{sp}(a)$. Set $I = \overline{(a - \lambda)A}$. Then I is clearly a closed ideal of A. No element of the form $(a - \lambda)b$ is invertible in the commutative Banach algebra A. By 1.1.10,

$$\|(a - \lambda)b - 1\| \ge 1.$$

So $1 \notin I$ and I is proper. Therefore, if A is simple, a must be a scalar, whence $A = \mathbb{C}$. □

Lemma 1.1.14 *If p is a polynomial and a is an element of a unital Banach algebra A, then*

$$\mathrm{sp}(p(a)) = p(\mathrm{sp}(a)).$$

Proof. We may assume that p is not constant. If $\lambda \in \mathbb{C}$, there are $c, \beta_1, ..., \beta_n \in \mathbb{C}$ such that

$$p(z) - \lambda = c\prod_{i=1}^{n}(z - \beta_i),$$

and therefore

$$p(a) - \lambda = c\prod_{i=1}^{n}(a - \beta_i).$$

It is clear that $p(a) - \lambda$ is invertible if and only if each $a - \beta_i$ is. It follows that $\lambda \in \mathrm{sp}(p(a))$ if and only if $\lambda = p(\alpha)$ for some $\alpha \in \mathrm{sp}(a)$. Thus $\mathrm{sp}(p(a)) = p(\mathrm{sp}(a))$. □

Definition 1.1.15 Let A be a unital Banach algebra. If $a \in A$, its *spectral radius* is defined to be

$$r(a) = \sup_{\lambda \in \mathrm{sp}(a)} |\lambda|.$$

Theorem 1.1.16 *If a is an element in a unital Banach algebra A, then*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Proof. If $\lambda \in \mathrm{sp}(a)$, then $\lambda^n \in \mathrm{sp}(a^n)$ by 1.1.14, so $|\lambda^n| \leq \|a^n\|$. Therefore, $r(a) \leq \liminf_{n \to \infty} \|a^n\|^{1/n}$. Let Ω be the open disk in $\mathbb{C}$ with center 0 and radius $1/r(a)$ (or ∞ if $r(a) = 0$). If $\lambda \in \Omega$, then $1 - \lambda a \in GL(A)$. If $f \in A^*$, then $f((1 - \lambda a)^{-1})$ is analytic. There are unique complex numbers z_n such that

$$f((1 - \lambda a)^{-1}) = \sum_{n=0}^{\infty} z_n \lambda^n \quad (\lambda \in \Omega).$$

However, if $|\lambda| < 1/\|a\| \leq 1/r(a)$, then $\|\lambda a\| < 1$, so $(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$, and therefore, $f((1 - \lambda a)^{-1}) = \sum_{n=0}^{\infty} \lambda^n f(a^n)$. It follows that $z_n = f(a^n)$ for all $n \geq 0$. Hence the sequence $\{\lambda^n f(a^n)\}$ converges to zero for each $\lambda \in \Omega$, and therefore is bounded. Since this is true for every $f \in A^*$, by the principle of uniform boundedness, $\{\lambda^n a^n\}$ is a bounded sequence. So we may assume that $|\lambda^n|\|a^n\| \leq M$ for all $n \geq 0$ and for some positive number M. Hence

$$\|a^n\|^{1/n} \leq \frac{M^{1/n}}{|\lambda|}, \quad n = 0, 1, ...$$

Consequently,

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le 1/|\lambda|.$$

This implies that if $r(a) < \frac{1}{|\lambda|}$, then $\limsup_{n\to\infty} \|a^n\|^{1/n} \le 1/|\lambda|$. It follows that $\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a)$. From what we have shown at the beginning of this proof, we obtain

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a) \le \liminf_{n\to\infty} \|a^n\|^{1/n}.$$

This implies that $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. □

Example 1.1.17 Let $A = M_3$ and $a = \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$. Then $\|a\| \ge 1$ and $\mathrm{sp}(a) = \{1/2, 1/3\}$. So $r(a) = 1/2$. It follows that $\|a^n\|^{1/n} \to 1/2$.

Let $T \in B(L^2([0,1]))$ be a bounded linear operator defined by

$$T(f) = \int_0^t f(x)dx.$$

The reader can compute that $\|T^n\| \le \frac{1}{n!}$. Hence $r(T) = 0$. Note that $T \ne 0$ (see Exercise 1.11.3).

We now establish the holomorphic functional calculus for elements in Banach algebras (1.1.19). The first application appears in 1.2.9. Later, we will establish continuous functional calculus for commutative C^*-algebras (1.3.5) and Borel functional calculus for normal elements in von Neumann algebras (1.8.5).

Definition 1.1.18 Let x be a fixed element in a unital Banach algebra A. Let f be a holomorphic function in an open neighborhood O_f of $\mathrm{sp}(x)$, and C be a smooth simple closed curve in O_f enclosing $\mathrm{sp}(x)$. We assign the positive orientation to C as in complex analysis. For each $\phi \in A^*$, we consider a continuous function which maps λ to $f(\lambda)\phi((\lambda - x)^{-1})$ on the curve C. Set

$$L(\phi) = \frac{1}{2\pi i} \int_C f(\lambda)\phi((\lambda - x)^{-1})d\lambda.$$

The map $\phi \mapsto L(\phi)$ is a linear functional on A^* and

$$|L(\phi)| \le \frac{l}{2\pi} \|\phi\| \sup\{|f(\lambda)| \|(\lambda - x)^{-1}\| : \lambda \in C\},$$

where l is the length of the curve C. Hence there exists an $F \in A^{**}$ such that $F(\phi) = L(\phi)$.

On the other hand, the function $\lambda \mapsto f(\lambda)(\lambda - x)^{-1}$ is a continuous function from C into A. So the limit of

$$\sum_{i=0}^{n} f(\lambda_i)(\lambda_i - x)^{-1}(\lambda_i - \lambda_{i+1}) \ \ (\text{as } \max_i|\lambda_i - \lambda_{i+1}| \to 0),$$

where $\{\lambda_0, ..., \lambda_n, \lambda_{n+1} = \lambda_0\}$ is a partition of the curve C, converges in norm in A to y. By the continuity of ϕ, we know that $\phi(y) = L(\phi) = F(\phi)$ for all $\phi \in A^*$. Hence $F \in A$. By Cauchy's theorem, F does not depend on the choice of the curve C, but only on the function f. Therefore we may denote F by $f(x)$ and write

$$f(x) = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda - x)^{-1}d\lambda.$$

We denote by $\mathrm{Hol}(\mathrm{sp}(x))$ the algebra of all functions which are holomorphic in a neighborhood of $\mathrm{sp}(x)$.

Theorem 1.1.19 *Fix an element x in a unital Banach algebra A. The map $f \mapsto f(x)$ from* $\mathrm{Hol}(\mathrm{sp}(x))$ *into A is a homomorphism which sends the constant function 1 to the identity of A and the identity function on $\mathbb{C}$ to the element x. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in a neighborhood of* $\mathrm{sp}(x)$, *then $f(x) = \sum_{n=0}^{\infty} c_n x^n$.*

Proof. Linearity is clear. Let f and g be functions holomorphic in neighborhoods O_f and O_g of $\mathrm{sp}(x)$, respectively. Set $O = O_f \cap O_g$ and let $C_i, i = 1, 2$ be smooth simple closed curves in O enclosing $\mathrm{sp}(x)$ such that C_1 lies completely inside the curve C_2. Then

$$\begin{aligned} f(x)g(x) &= \left(\frac{1}{2\pi i}\int_{C_1} f(\lambda)(\lambda - x)^{-1}d\lambda\right)\left(\frac{1}{2\pi i}\int_{C_2} g(z)(z - x)^{-1}dz\right) \\ &= -\frac{1}{4\pi^2}\int_{C_1}\left[\int_{C_2} f(\lambda)g(z)(\lambda - x)^{-1}(z - x)^{-1}dz\right]d\lambda \\ &= -\frac{1}{4\pi^2}\int_{C_1}\int_{C_2} f(\lambda)g(z)\frac{[(z - x)^{-1} - (\lambda - x)^{-1}]}{\lambda - z}dzd\lambda \\ &= \frac{1}{2\pi i}\int_{C_1} f(\lambda)(\lambda - x)^{-1}\left(\frac{1}{2\pi i}\int_{C_2}\frac{g(z)}{z - \lambda}dz\right)d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{C_2} g(z)(z - x)^{-1}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(\lambda)}{\lambda - z}d\lambda\right)dz \end{aligned}$$

$$= \frac{1}{2\pi i}\int_{C_1} f(\lambda)g(\lambda)(\lambda - x)^{-1}d\lambda = (f\cdot g)(x).$$

The second to last equality holds because $\frac{1}{2\pi i}\int_{C_2}\frac{g(z)}{z-\lambda}dz = g(\lambda)$ (by the Cauchy formula) and because $\frac{f(\lambda)}{\lambda - z}$ is holomorphic inside the curve C_2 if $\lambda \in C_1$ (so that $\int_{C_1}\frac{f(\lambda)}{\lambda-z}d\lambda = 0$).

To complete the proof, pick a circle C with center at 0 and large radius. We note that

$$\lambda^{n-1}(1-\lambda^{-1}x)^{-1} = \lambda^{n-1}\sum_{n=0}^{\infty} x^k\lambda^{-k} = \sum_{k=0}^{\infty} x^k\lambda^{n-k-1},$$

where the convergence is in norm and is uniform on C. Therefore

$$\int_C x^k\lambda^{n-k-1}d\lambda = (\int_C \lambda^{n-k-1}d\lambda)x^k = 0\cdot x^k = 0 \qquad \text{(e 1.2)}$$

unless $k = n$, in which case the integral is $2\pi i x^n$. This implies that

$$x^n = \frac{1}{2\pi i}\int_C \lambda^n(\lambda - x)^{-1}d\lambda. \qquad \text{(e 1.3)}$$

Now suppose that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in a neighborhood of $\mathrm{sp}(x)$. Then it converges in an open disk with center 0. Let C be a circle with center at 0 contained in the open disk. Then the series converges uniformly on C, so from (e 1.3),

$$f(x) = \frac{1}{2\pi i}\int_C f(z)(z-x)^{-1}dz = \sum_{n=0}^{\infty} c_n(\frac{1}{2\pi i}\int_C z^n(z-x)^{-1}dz) = \sum_{n=0}^{\infty} c_n x^n.$$ □

Proposition 1.1.20 *If I is a closed ideal in a Banach algebra, then A/I is a Banach algebra with the quotient norm*

$$\|\bar{a}\| = \|a + I\| = \inf_{b\in I}\|a+b\|.$$

Proof. It is well known that A/I, as a normed space is complete. It remains to show that A/I is a normed algebra. Let $\varepsilon > 0$ and $a, b \in A$. Then, there are $i, j \in I$ such that

$$\|a+i\| < \|a+I\| + \varepsilon \text{ and } \|b+j\| < \|b+I\| + \varepsilon.$$

Hence, for $c = ib + aj + ij \in I$,

$$\|ab + c\| \le \|a+i\|\|b+j\| < (\|a+I\|+\varepsilon)(\|b+I\|+\varepsilon).$$