Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zurich

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Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory

Springer-Verlag
Berlin · Heidelberg · New York 1973

AMS Subject Classifications (1970): Primary: 40J05, 41A25
Secondary: 26A33, 40D15, 40G05, 42A56

ISBN 3-540-06357-9 Springer-Verlag Berlin · Heidelberg · New York
ISBN 0-387-06357-9 Springer-Verlag New York · Heidelberg · Berlin

Offsetdruck: Julius Beltz, Hemsbach/Bergstr.

PREFACE

In recent years some of the fundamental problems of approximation theory have turned out to be the verification of Jackson-, Bernstein-, and Zamansky-type inequalities for particular approximation processes, a study of the comparison of two different processes with respect to their rate of convergence, as well as the associated problems of non-optimal and optimal (or saturated) approximation for given processes.

These problems are here examined in the frame of abstract Fourier series in Banach spaces with respect to a total, fundamental sequence of mutually orthogonal projections $\{P_k\}$, the approximation processes being of multiplier type - i.e. (in some sense) summation methods of the abstract series.

In view of the multiplier structure it turns out that problems such as those mentioned above may be transferred to corresponding ones upon the coefficients (associated to the approximation processes in question) in the form of uniform multiplier conditions. In order to check such conditions multiplier criteria are required. To develop such, by the applications in mind it is most convenient to assume the uniform boundedness of the Cesàro means of order α (for some $\alpha \geq 0$) of the abstract expansion. Fractional α are admitted since the applications are to cover, for example, the Riesz means of classical trigonometric series expansions. Now the mere hypothesis that the (C,α)- means are bounded suffices to develop a most useful multiplier theory for the Banach space X with respect to the projections $\{P_k\}$.

In particular, it turns out that each scalar sequence $\eta = \{\eta_k\}_{k\in \mathbf{P}}$ ($\mathbf{P}$ the set of non-negative integers) belonging to the set $bv_{\alpha+1}$ is a multiplier sequence; here the $bv_{\alpha+1}$-norm of η is some sum of (fractional) differences of η. Since, in general, it is difficult to check whether $\eta \in bv_{\alpha+1}$ or not, η on $\mathbf{P}$ is suitably extended to a function e(x) defined on $\mathbf{R}_1^+$ belonging to a suitable set $BV_{\alpha+1}$ (BV_1 is the set of functions of bounded variation). Then one has the fundamental inclusions

$$(*) \qquad BV_{\alpha+1}|_{\mathbf{P}} \subset bv_{\alpha+1} \subset M.$$

The methods employed involve the theories of fractional differences and differentiation, many results of which the author first had to polish up. In particular, results of classical summability theory such as those of A.F. Andersen 1928 (in connection with the set $bv_{\alpha+1}$) and of G.H. Hardy 1916 (Second theorem of consistency) were suitably modified. With respect to the theory of fractional differentiation, results of H. Weyl 1917 (Lipschitz conditions and the existence of fractional derivatives), J.J. Gergen 1937 (connection between sums over fractional differences and integrals over fractional derivatives) as well as J. Cossar 1941 (definition of a fractional derivative) could be used or carried over.

The multiplier theory is so sharp that relation (*) is necessary and sufficient for the Cesàro means of the same order to be uniformly bounded. On the other hand, this multiplier theory is quite useful (also in the strictly fractional case) as is shown by several examples such as the means of Abel-Cartwright, Riesz, Picard, Cesàro, de La Vallée-Poussin. For these approximation processes the series of problems mentioned above is solved in Banach spaces, provided only that the expansions in X with respect to $\{P_k\}$ have uniformly bounded Cesàro means of some order.

Finally, particular choices of the Banach space and its sequence of projections yield new and deep applications to one - and multiple trigonometric series, to Laguerre and Hermite series, as well as to expansions into Jacobi polynomials or spherical harmonics. For all these expansions (and those which are (C,α)-bounded) this elegant and lucid approach gives a multiplier theory together with a large number of new approximation theoretical results and reveals the outstanding role of Cesàro (or Riesz) summability.

Summarizing, let me emphasize that one of the major contributions of Trebels' work is the fact that he seems to have been the first to apply the theory of divergent series (in its vector-valued form) - a theory consisting of "hard analysis" that took about three centuries to develop - to the more modern approximation theory. Furthermore, he clearly pointed out an intimate connection (still to be established in its full generality) between summability theory of abstract orthogonal expansions and multiplier theory. Sharper results in summability theory should lead to sharper results in multiplier theory (and conversely), and these in turn will lead to a sharper approximation theory - a most promising and interesting research problem. However, concerning (C,α)-bounded expansions, this problem has been solved and is to be found in this contribution. It also delivers in a unified presentation very practical multiplier theories for Laguerre, Hermite and Jacobi series, which seem to be new. The present article, which has been written in form of a monograph, should receive a wide audience.

Aachen, January 1973

P.L. Butzer

CONTENTS

1. INTRODUCTION

1.1 General background

The origin of the present investigation is to be seen in two papers by Butzer - Nessel - Trebels [33; I,II] concerned with two problems of Favard [43, 44], namely on the comparison of summation processes of abstract Fourier series in a Banach space X (with respect to their rate of convergence) and on the saturation problem. Whilst treating these problems it turned out that there would be needed a suitable multiplier theory. In order to develop one, and in view of the applications in mind (to Fourier, ultraspherical, Laguerre series, and so on), it was convenient to assume the Cesàro means of order j ($j \in \mathbf{P}$, the set of non-negative integers) of the abstract Fourier expansion to be uniformly bounded. The sufficient multiplier criterium resulting from this is the classical one for numerical series summable (C,j) (cf.[61; p.128]).

The parallelism existing between multiplier criteria for numerical (C,α)-bounded series ($\alpha \geq 0$) and for classical Fourier series has already been observed by Moore [75] and Goes [53, 54] in their investigations on sufficient multiplier criteria for one-dimensional, trigonometric series expansions on several particular function spaces with the aid of suitable sums of fractional differences; for further literature see [52].

Apart from the abstract framework, the following idea (cf.[33]) is a decisive advantage with respect to applications, namely to extend suitably the multiplier sequence $\eta = \{\eta_k\}$, defined on $\mathbf{P}$, to a function $e(x)$ defined on the positive half-axis ($x \geq 0$) and then estimate the sum (over the differences of η) by a suitable integral over derivatives of the function $e(x)$. (By the way, this procedure is already implicitly contained in [61; p.373]).

However, though in [33] α is restricted to $\mathbf{P}$ the example of the Riesz-means (see (3.35)) of classical trigonometric series expansions (for small exponents λ) calls in [33] for an extension of the theory to all $\alpha \geq 0$. This will be performed in Section 3, where an analog of Hardy's [60] "Second theorem of consistency" is added, which will considerably simplify the computations in the applications. Let us remark that the resulting integral criterium, i.e., the extension of the above mentioned criterium (cf.[61; p.373]) to all $\alpha \geq 0$ is almost identical to criteria for numerical series by Borwein [20], Maddox [71], Russel [86] and others, though their context as well as their methods seem to be quite different.

Our approach is based immediately upon the hypothesis that the (C,α)-means of the abstract expansion is uniformly bounded. We essentially use Gergen's [50] elegant proof on the equivalence of Cesàro and Riesz summability of numerical series which can be carried over to our situation without difficulties. (The corresponding proof of Ingham [65] will probably also work; for α an integer this is obvious by [61; p.113]).

Since many results on (C,α)-boundedness in norm of concrete orthogonal expansions have only been proved in the last ten years, there was no great demand for a unified approach to norm estimates in the past, and there seem to be only a few papers using the parallelism between the multiplier theory of classical summability theory and of expansions in Banach spaces (one of the first seems to have been Hille [63]). On the other hand, for pointwise results this correspondence has been extensively used in case of Dirichlet, Fourier, power series, and so on (see e.g. the books [39; Ch.3,4], [61], [108; p.154], and only recently [97; p.285]).

A systematic exploitation of this parallelism will give a number of analogous results for various Banach-valued expansions, and not only (C,α)-bounded ones, as well as results for operators from X into another space (for one-dimensional trigonometric series see e.g. [53, 54]).

But here we restrict ourselves to multiplier theory for (C,α)-bounded, Banach-valued expansions in X, and to its application to several (norm-) approximation problems.

Before sketching the latter ones let us mention the "transplantation" approach as given axiomatically for series expansions by Gilbert [51] which generalizes basic work of Askey-Wainger [8] and Askey [4]. This approach is based heavily upon Riesz' theorem that the partial sums (i.e. $(C,0)$-means) of the classical one-dimensional Fourier series converge in norm for $1 < p < \infty$. It allows one to associate the trigonometric series with an expansion in functions $\{u_k\}$ defined on $(0,\pi)$ (not necessarily orthogonal), which are "similar" to $\{\cos kx\}$ and {sin kx} (e.g. perturbed cosine and sine functions, Fourier-Bessel and Fourier-Dini functions, Jacobi polynomials, eigenfunctions of fairly general Sturm-Liouville problems and so on - see [51]). Then the Abel means of the latter expansion are uniformly bounded in L^p; using the Marcinkiewicz multiplier theorem for the trigonometric system in weighted L^p-spaces one can obtain its analog for the system $\{u_k\}$ (further restrictions on the domain of p seem to be involved, see [8], [4]).

The advantages of the "transplantation" approach are the following: i) one does not need any knowledge on (C,α)-boundedness of the expansion, ii) the system need not necessarily be orthogonal, and iii) the sharp known results for the trigonometric system may be applied.

The advantages of our approach via the (C,α)-means are to be seen in: i) it is not restricted to functions similar to {cos kx}, {sin kx}, ii) multiplier criteria may be derived in an elementary and direct manner on all L^p-spaces from the (C,α)-boundedness hypothesis.

Now, for the sake of completeness, let us briefly sketch the approximation problems for which we wish to apply the multiplier theory mentioned above.

1.2 Approximation theory in Banach spaces

Generalizing classical approximation theorems of Jackson, Bernstein, Zamansky, etc., Butzer-Scherer [37] have shown that in any Banach space X there holds a general approximation theorem for linear approximation processes (with respect to their rate of convergence) provided only that Jackson- and Bernstein-type inequalities are satisfied for these processes.

To give a few details concerning this theory let X be a Banach space with norm $\|\cdot\|$, and let [X] be the set of all bounded linear operators from X into X. Consider a family $\mathcal{T} = \{T(\varepsilon);\ \varepsilon \in [0,1]\}$ of strongly measurable operators in [X] satisfying

$$(1.1) \quad T(0) = I, \quad T(\varepsilon_1)T(\varepsilon_2) = T(\varepsilon_2)T(\varepsilon_1) \qquad (\varepsilon_1,\varepsilon_2 \in [0,1]),$$

$$(1.2) \quad \|T(\varepsilon)f\| \leq M_{\mathcal{T}}\|f\| \;, \qquad \lim_{\varepsilon\to 0+} \|T(\varepsilon)f-f\| = 0 \qquad (f \in X);$$

further introduce a Banach subspace Y of X with semi-norm $|\cdot|_Y$ and norm $\|\cdot\|_Y = |\cdot|_Y + \|\cdot\|$ so that Y is continuously embedded in X, in notation $Y \subset X$. Then

Definition 1.1. *Let $Y \subset X$ and $y(\varepsilon)$ be a monotonely increasing function such that $0 < y(\varepsilon) \leq y(1) = 1$ and (m_y being a constant)*

$$(1.3) \qquad y(\varepsilon) \leq m_y \; y(\varepsilon/2) \qquad (\varepsilon \in (0,1]).$$

a) $\mathcal{T}$ is said to satisfy a Jackson-type inequality of order $y(\varepsilon)$ on X with respect to Y provided

$$\|T(\varepsilon)f-f\| \leq D_Y \; y(\varepsilon)|f|_Y \qquad (f \in Y)$$

for some constant D_Y (independent of f and ε).

b) $\mathcal{T}$ is said to satisfy a Bernstein-type inequality of order $y(\varepsilon)$ on X with respect to Y provided $T(\varepsilon)$ is strongly measurable on Y (in particular $T(\varepsilon)(X) \subset Y$) and

$$|T(\varepsilon)f|_Y \leq D_Y^*[y(\varepsilon)]^{-1}\|f\| \qquad (f \in X)$$

for some constant D_Y^*.

Now we may formulate a quite special case of the general approximation theorem of Butzer-Scherer [37; Cor.2] mentioned above:

Theorem 1.2. *Let $Y \subset Z \subset X$ and $\mathcal{T}$ be as above satisfying Jackson- as well as Bernstein-type inequalities of orders $y(\varepsilon)$ and $z(\varepsilon)$ on X with respect to Y and Z, respectively. Let $y(\varepsilon)$ and $z(\varepsilon)$ satisfy*

$$(1.4) \qquad \begin{aligned} &\int_0^{\varepsilon} y(u)[z(u)]^{-1}\, u^{-1}du = O\big(y(\varepsilon)[z(\varepsilon)]^{-1}\big) \\ &\int_{\varepsilon}^{1} [y(u)]^{-1}\, z(u)\; u^{-1}du = O\big([y(\varepsilon)]^{-1}\, z(\varepsilon)\big). \end{aligned}$$

Let $\Omega(\varepsilon)$ be a positive, nondecreasing function satisfying

(1.5)
$$\int_0^{\varepsilon} [z(u)]^{-1}\,\Omega(u)u^{-1}du = O\big([z(\varepsilon)]^{-1}\Omega(\varepsilon)\big),$$
$$\int_{\varepsilon}^{1} [y(u)]^{-1}\,\Omega(u)u^{-1}du = O\big([y(\varepsilon)]^{-1}\Omega(\varepsilon)\big).$$

Then the following assertions are equivalent for $\varepsilon\to 0+$:

(a) $\quad \|T(\varepsilon)f-f\| = O\big(\Omega(\varepsilon)\big),$

(b) $\quad |T(\varepsilon)f|_Y = O\big([y(\varepsilon)]^{-1}\Omega(\varepsilon)\big),$

(c) $\quad f\in Z,\ |T(\varepsilon)f-f|_Z = O\big([z(\varepsilon)]^{-1}\Omega(\varepsilon)\big),$

(d) $\quad K(y(\varepsilon),f;\,X,Y) \equiv \inf_{g\in Y}\big(\|f-g\| + y(\varepsilon)|f|_Y\big) = O\big(\Omega(\varepsilon)\big).$

If in addition Ω satisfies the further conditions

(1.6)
$$\int_{\varepsilon}^{1} z(u)[\Omega(u)]^{-1}u^{-1}du = O\big(z(\varepsilon)[\Omega(\varepsilon)]^{-1}\big),$$
$$\int_0^{\varepsilon} y(u)[\Omega(u)]^{-1}u^{-1}du = O\big(y(\varepsilon)[\Omega(\varepsilon)]^{-1}\big),$$

then the following assertions are equivalent for $1\le q<\infty$:

(a)' $\quad \int_0^1 \{[\Omega(u)]^{-1}\|T(u)f-f\|\}^q\,u^{-1}du < \infty ,$

(b)' $\quad \int_0^1 \{[\Omega(u)]^{-1}\,y(u)|T(u)f|_Y\}^q\,u^{-1}du < \infty,$

(c)' $\quad f\in Z,\ \int_0^1 \{[\Omega(u)]^{-1}\,z(u)|T(u)f-f|_Z\}^q\,u^{-1}du < \infty ,$

(d)' $\quad \int_0^1 \{[\Omega(u)]^{-1}\,K(y(u),\,f;X,Y)\}^q\,u^{-1}du < \infty .$

Let us remark that for the particular choice $\Omega(\varepsilon) = \varepsilon^{\beta}$, $y(\varepsilon) = \varepsilon^{m}$, and $z(\varepsilon) = \varepsilon^{k}$, the assumptions read $0 \le k < \beta < m$, and that in concrete spaces the K-functional (introduced by Peetre, cf.(d)) corresponds to a modulus of continuity (cf. [26; Ch.4]). For more general results we refer to [37], for applications to [35,36,37]. Let us only mention that there is also a discrete version (instead of $\{T(\varepsilon);\ 0 \leq \varepsilon \leq 1\}$ one has a countable family $\{S(n)\} \subset [X]$ satisfying (1.1) and (1.2)), in which case the measurability conditions simply reduce to $S(n)f \in Y$ and $S(n)f \in Z$, respectively. The choice of $0 \leq \varepsilon \leq 1$ is only technical and may be replaced by $\varepsilon^{-1} = \rho$, $\rho \geq 1$. Clearly one may take $\rho > 0$, and since this notation coincides with the standard one, we will use it henceforward.

For an application of Theorem 1.2 one essentially has to check whether convenient Jackson- and Bernstein-type inequalities are satisfied. This will be carried out in Sections 2, 4, 5 in connection with summation processes of Fourier expansions in Banach spaces (without reformulating Theorem 1.2 in concrete examples).

The saturation problem, mentioned at the beginning, may be interpreted as the problem of an optimal Jackson-type inequality. It was first introduced by Favard for summation methods of trigonometric series in a lecture in 1947 (cf.[43]) and may be formulated as follows (see e.g. [31; p.434]).

<u>*Definition 1.3.*</u> *The strong approximation process* $\mathcal{T} = \{T(\rho);\ \rho > 0\}$ *(cf. (1.2)) is said to possess the saturation property if there exists a positive function* $\Theta(\rho)$, $\rho > 0$, *tending monotonely to infinity as* $\rho \to \infty$ *such that every* $f \in X$ *for which*

$$\|\Theta(\rho)[T(\rho)f-f]\| = o(1) \qquad (\rho \to \infty)$$

is an invariant element of $\mathcal{T}$, *i.e.* $T(\rho)f = f$ *for all* $\rho > 0$, *and if the set*

$$F[X;\mathcal{T}] = \{f \in X;\ \|\Theta(\rho)[T(\rho)f-f]\| = O(1),\ \rho \to \infty\}$$

contains at least one noninvariant element. In this event, the approximation process $\mathcal{T}$ *is said to have optimal approximation order* $[\Theta(\rho)]^{-1}$ *or to be saturated in* X *with order* $[\Theta(\rho)]^{-1}$, *and* $F[X;\mathcal{T}]$ *is called its Favard or saturation class.*

Today there exists a vast literature concerned with saturation for various types of approximation processes. To mention general approaches in regard to solution, there exists an integral transform method in diverse Lebesgue spaces as well as the semi-group method on arbitrary Banach spaces in its extended form (for detailed bibliographical comments one may consult the books of Berens [15], Butzer-Berens [26], and Butzer-Nessel [31]).

The implication (a) $\Longrightarrow$ (b) in Theorem 1.2 is called a Zamansky-type inequality which, however, suffers under the restrictions (1.5) and (1.6) upon Ω. In case $[y(\varepsilon)]^{-1} = \Theta(\rho)$ is the saturation order and the relative completion of Y is $F[X; \mathcal{T}]$ (see Def. 2.5), this inequality may be established without the above restrictions via the direct estimate

$$(1.7) \qquad |T(\rho)f|_Y \leq D\,\Theta(\rho)\|T(\rho)f-f\| \qquad (f \in X).$$

These matters, as well as extensions, will be treated in Sections 2, 4, 5 for approximation processes given via Fourier expansions in Banach spaces.

Let us finally introduce the comparison problem for two summation methods, mentioned at the beginning and posed by Favard [44].

<u>*Definition 1.4.*</u> *Let $\mathcal{S}$ and $\mathcal{T}$ be two approximation processes on X satisfying (1.2). $\mathcal{T}$ is said to be better than $\mathcal{S}$ (with respect to its rate of convergence), if there exists a constant D > 0 such that*

$$(1.8) \qquad \|T(\rho)f-f\| \leq D\|S(\rho)f-f\| \qquad (f \in X;\ \rho > 0).$$

If $\mathcal{T}$ is better than $\mathcal{S}$ and the latter in turn better than $\mathcal{T}$, then the processes are said to be equivalent, in notation

$$\|T(\rho)f-f\| \approx \|S(\rho)f-f\| \qquad (f \in X).$$

First contributions to this problem have been made by Shapiro [89], Boman-Shapiro [19], and Butzer-Nessel-Trebels [32, 33;I] (compare the comments in [31; p.507], [32]). Whereas in [89], [19] (cf. Löfström [70] for precursory material) the concrete case of approximation processes representable as Fourier convolution integrals of Fejér's type is considered for Euclidean n-space (or n-dimensional torus), in [32, 33;I] the problem is discussed in the setting of abstract Hilbert spaces and of expansions in Banach spaces, respectively.

We dispense with a survey of the following sections and refer to the short summary preceding each section.

Acknowledgements

These investigations were carried out under a "DFG-Habilitanden-stipendium" and the author is much obliged to the DFG for its generous support. He is very much indebted to Professor P.L. Butzer for his continuous encouragement and promotion. It was he who suggested the problem which was developped in cooperation with him and Professor R.J. Nessel; the two papers [33; I,II] reveal their decisive contributions; for this and their critical reading he is profoundly grateful. He also acknowledges with pleasure the numerous helps and hints given in various discussions by his colleagues, in particular by Doz. Dr. E. Görlich and Drs. H. Johnen, K. Scherer, E.L. Stark and U. Westphal

2. GENERAL THEORY

In this section the concept of abstract expansions in Banach spaces X is introduced with the aid of a countable family of projections $\{P_k\}$, and multipliers with respect to a fixed pair $X, \{P_k\}$ are studied. Restricting the approximation processes to those of multiplier type (i.e. some kind of "convolution" structure is assumed), sufficient conditions for Jackson-, Bernstein-, Zamansky-type inequalities as well as for saturation and comparison theorems are formulated in terms of multipliers. The framework as well as the comparison theorem is taken over from [33;I]; for the saturation theorem see [33;II], for the Bernstein-type inequality Görlich-Nessel-Trebels [59].

2.1 Notations and further definitions

As in Sec. 1.2, let X be an arbitrary (real or complex) Banach space with norm $\|\cdot\|$ and elements f, g,...; let [X] be the Banach algebra of all bounded linear operators on X into itself. Further, denote by $\mathbb{R}$ and $\mathbb{C}$ the set of all real and complex numbers, respectively, and let $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ be the sets of all, of all non-negative, of all positive integers, respectively. By $[\alpha]$ denote the largest integer less than or equal to $\alpha \in \mathbb{R}$. Let us decompose the Banach space X by a sequence of projections $\{P_k\}_{k \in \mathbb{P}} \subset [X]$ satisfying the following properties

i) the projections P_k are mutually orthogonal, i.e., for all $j, k \in \mathbb{P}$ there holds $P_j P_k = \delta_{j,k} P_k$, $\delta_{j,k}$ being Kronecker's symbol;

ii) the sequence $\{P_k\}$ is total, i.e., $P_k f = 0$ for all $k \in \mathbb{P}$ implies $f = 0$;

iii) the sequence $\{P_k\}$ is fundamental[1)], i.e., the linear span of the ranges $P_k(X)$, $k \in \mathbb{P}$, is dense in X: $\overline{\bigcup_{k \in \mathbb{P}} P_k(X)} = X$.

1) On account of the Banach-Steinhaus theorem this property is a necessary one for a uniformly bounded family of operators $\{T(\rho)\}$ to imply convergence on all X provided $T(\rho)$ converges on each $P_k(X)$, $k \in \mathbb{P}$. However, it is irrelevant for the multiplier criteria in Sec.3 .

Then with each $f \in X$ one may associate its (formal) Fourier series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f \qquad (f \in X). \tag{2.1}$$

With s the set of all sequences $\eta = \{\eta_k\}_{k\in \mathbb{P}}$ of scalars, $\eta \in s$ is called a multiplier for X (corresponding to $\{P_k\}$), if for each $f \in X$ there exists an element $f^\eta \in X$ such that $\eta_k P_k f = P_k f^\eta$ for all $k \in \mathbb{P}$, thus

$$f^\eta \sim \sum_{k=0}^{\infty} \eta_k P_k f . \tag{2.2}$$

Note that f^η is uniquely determined by f since $\{P_k\}$ is total. The set of all multipliers is denoted by $M = M(X;\{P_k\})$. With the natural vector operations, coordinatewise multiplication and norm

$$\|\eta\|_M = \sup\{\|f^\eta\| ;\ f \in X,\ \|f\| \leq 1\}, \tag{2.3}$$

M is a commutative Banach algebra containing the identity $\{1\} \in s$.

An operator T from X into itself is called a multiplier operator if there exists a sequence $\tau \in s$ such that $P_k Tf = \tau_k P_k f$ for all $f \in X$, $k \in \mathbb{P}$, i.e., one has the formal expansion

$$Tf \sim \sum_{k=0}^{\infty} \tau_k P_k f \qquad (f \in X). \tag{2.4}$$

Obviously, $T \in [X]$. Thus, by definition, with each multiplier operator $T \in [X]_M$ (the set of all multiplier operators on X) there is associated a multiplier sequence $\tau \in M$ and vice versa, and since $\|T\|_{[X]} = \|\tau\|_M$ by definition (cf. (2.3)), M can be identified with $[X]_M$. In the future we always assume $\mathcal{T}, \mathcal{J} \subset [X]_M$.

Remark. The expansion (2.1) represents a slight generalization of the concept of Fourier series in a Banach space X associated with a fundamental, total, biorthogonal system $\{f_k, f_k^*\}$. Here $\{f_k, f_k^*\}$ consists of two sequences $\{f_k\} \subset X, \{f_k^*\} \subset X^*$ such that i) $f_j^*(f_k) = \delta_{j,k}$ for all $j,k \in \mathbb{P}$ (orthogonal), ii) $f_k^*(f) = 0$ for all $k \in \mathbb{P}$ implies $f = 0$ (total), and iii) the linear span of $\{f_k\}$ is dense in X (fundamental).

Then (2.1) and (2.4) read

$$f \sim \sum_{k=o}^{\infty} f_k^*(f) f_k, \quad Tf \sim \sum_{k=o}^{\infty} \tau_k f_k^*(f) f_k \ ,$$

respectively.

For these definitions and results compare Marti [72;p.86 ff], Milman [73] , Singer [91;pp.1-49] , etc.

In this framework, the general approximation Theorem 1.2 of Butzer-Scherer [37] suggests that one determines subspaces of X via some sequences of s which do not necessarily belong to M. For arbitrary $\psi \in s$ we define

$$(2.5) \qquad X^{\psi} = \{f \in X; \exists\, f^{\psi} \in X \text{ with } \psi_k P_k f = P_k f^{\psi} \text{ for all } k \in \mathbb{P} \} \ .$$

Obviously, if B^{ψ} is the operator with domain X^{ψ} and range in X defined by $B^{\psi} f = f^{\psi}$, $f \in X^{\psi}$, then B^{ψ} is a closed linear operator for each $\psi \in s$. Since $P_k(X)$ is contained in X^{ψ} for each $k \in \mathbb{P}$, B^{ψ} is densely defined. Further, defining a semi-norm on X^{ψ} via $|f|_{\psi} = \|B^{\psi} f\|$, X^{ψ} is a Banach space with respect to the norm $\|f\| + |f|_{\psi}$, and $X^{\psi} \subset X$.

2.2 Jackson- and Bernstein-type inequalities

The first general result, in fact just a reformulation in the present setting, reads (cf. [33, 59])

Theorem 2.1. *Let $\mathcal{T} \subset [X]_M$ be a strong approximation process with associated multiplier family $\{\tau(\rho)\}_{\rho>o}$.*

a) If there exist a non-negative, monotonely increasing function $\chi(\rho)$ with $\lim_{\rho\to\infty}\chi(\rho) = \infty$, $\psi \in s$, and a uniformly bounded multiplier family $\{\eta(\rho)\} \subset M$ with

$$(2.6) \qquad \chi(\rho)\{\tau_k(\rho) - 1\} = \psi_k \eta_k(\rho),$$

then one has the Jackson-type inequality

$$\chi(\rho)\|T(\rho)f-f\| \leq \sup_{\rho>0} \|\eta(\rho)\|_M |f|_\psi \qquad (f \in X^\psi). \tag{2.7}$$

b) If (2.6) is replaced in a) by

$$\psi_k \tau_k(\rho) = \chi(\rho)\eta_k(\rho) , \tag{2.8}$$

then there holds the Bernstein-type inequality

$$|T(\rho)f|_\psi \leq \chi(\rho) \sup_{\rho>0}\|\eta(\rho)\|_M \|f\| \qquad (f \in X). \tag{2.9}$$

The proofs of a) and b) immediately follow by the hypotheses. For example, since $\{P_k\}$ is total, (2.6) is equivalent to

$$\chi(\rho)\{T(\rho)f-f\} = E(\rho)(B^\psi f),$$

where $E(\rho)$ is the (uniformly bounded) operator associated to the (uniformly bounded) multiplier $\eta(\rho)$; hence (2.7) holds.

This theorem induces one to expect that the verification of multiplier conditions (such as (2.6) and (2.8)) will present the actual problem, and Section 3 is therefore devoted to establishing convenient criteria concerning multipliers.

2.3 A saturation theorem

Let $\mathsf{K} = \{k \in \mathsf{P} ; \tau_k(\rho) = 1 \text{ for all } \rho > 0\}$ and assume $\mathsf{K} \neq \mathsf{P}$. Then the following condition upon $\mathcal{T}$ ensures the saturation property.

Definition 2.2. *The approximation process $\mathcal{T} \subset [X]_M$ satisfies condition (F), if there exist $\varphi \in s$ with $\varphi_k \neq 0$ for $k \notin \mathsf{K}$ and a non-negative, monotonely increasing function $\Theta(\rho)$ with $\lim_{\rho\to\infty} \Theta(\rho) = \infty$ such that*

$$\lim_{\rho\to\infty} \Theta(\rho)\{\tau_k(\rho) - 1\} = \varphi_k \qquad (k \in \mathsf{P}) \tag{2.10}$$

Condition[2)] (F) is a standard one in the study of saturation for summation processes of trigonometric series (cf. [31;p.435]). In fact, it was already introduced by Favard [43] in connection with fundamental, total biorthogonal systems (cf. Remark in Sec. 2.1) in arbitrary Banach spaces. As a consequence, the following result is substantially contained in [43] (the formulation is taken over precisely from [33;II]).

Lemma 2.3. *Let $f \in X$ and $\mathcal{T}$ satisfy condition (F).*

a) If there exists $g \in X$ such that

$$\lim_{\rho\to\infty} \|\Theta(\rho)\{T(\rho)f-f\} - g\| = 0 ,$$

the Fourier expansion of g is given by $g \sim \sum_{k=0}^{\infty} \varphi_k P_k f$.

b) $\Theta(\rho)\|T(\rho)f-f\| = o(1)$ implies $f \in \bigcup_{k\in K} P_k(X)$, and $T(\rho)f = f$ for all $\rho > 0$, thus f is an invariant element.

c) There exists some noninvariant $h \in X$ with $\Theta(\rho)\|T(\rho)h-h\| = O(1)$.

Proof. a) Since $P_k \in [X]$ and

$$P_k(\Theta(\rho)\{T(\rho)f-f\}) = \Theta(\rho)\{\tau_k(\rho)-1\} P_k f$$

one has for each $k \in P$

$$\|\varphi_k P_k f - P_k g\| = \lim_{\rho\to\infty} \|\Theta(\rho)\{\tau_k(\rho) - 1\}P_k f - P_k g\|$$

$$\leq \lim_{\rho\to\infty} \|P_k\|_{[X]} \|\Theta(\rho)\{T(\rho)f-f\} - g\| = 0 ,$$

which proves the assertion.

b) With g = 0 part a) gives $\varphi_k P_k f = 0$ for all $k \in P$. In case $k \notin K$ it follows that $P_k f = 0$, whereas for $k \in K$ the normalization $\tau_k(\rho) = 1$

2) Note that in (2.6) - (2.9) the choice of ψ and χ is variable, whereas φ and Θ in (2.10) and (2.11) are determined by the process.

for all $\rho > 0$ gives $P_k T(\rho) f = P_k f$. Thus $P_k T(\rho) f = P_k f$ for all $k \in \mathsf{P}$, and since $\{P_k\}$ is total, the assertion follows.

c) Since for any $h \in P_k(X)$

$$\| T(\rho)h - h \| = |\tau_k(\rho) - 1| \, \| h \| ,$$

$h \neq 0$ is noninvariant if $k \notin \mathsf{K}$, and the assertion follows by condition (F).

<u>*Definition 2.4.*</u> *The approximation process $\mathcal{T} \subset [X]_M$ is said to satisfy condition (F*), if (F) holds and there exists a uniformly bounded multiplier family $\{\eta(\rho)\} \subset M$ such that*

$$\Theta(\rho)\{\tau_k(\rho) - 1\} = \eta_k(\rho)\varphi_k \qquad (k \in \mathsf{P} \ , \rho > 0). \tag{2.11}$$

Condition (F^*) is also standard in saturation theory (cf. [31;Sec. 12.6] for detailed comments). Certainly, (F^*) (in connection with (F)) implies $\lim_{\rho\to\infty} \eta_k(\rho) = 1$ (it is assumed on K) so that by the theorem of Banach-Steinhaus the family $\{E(\rho)\}$ of operators corresponding to $\{\eta(\rho)\}$ forms a strong approximation process ($\{P_k\}$ is fundamental) satisfying $E(\rho)(X) \subset X^{\varphi}$ for all $\rho > 0$. Relation (2.11) immediately implies

$$\| \Theta(\rho)\{T(\rho)f - f\} \| = \| B^{\varphi} E(\rho) f \| \qquad (\rho > 0;\ f \in X) \tag{2.12}$$

and we have to discuss conditions upon f such that these expressions are uniformly bounded in ρ.

In this context, the idea of relative completion turns out to be fundamental (cf. Berens [15;p.14,p.28], [31;Sec.10.4]).

<u>*Definition 2.5.*</u> *Let $Y \subset X$ with semi-norm $|\cdot|_Y$. The completion of Y relative to X, denoted by $Y^{\sim X}$, is the set of those elements $f \in X$ for which there exists a sequence $\{f_n\} \subset Y$ and a constant $D > 0$ such that $|f_n|_Y \leq D$ for all n together with $\lim_{n\to\infty} \| f_n - f \| = 0$. With any $f \in Y^{\sim X}$ one may associate the semi-norm*

$$|f|_{Y^{\sim X}} = \inf\{\sup |f_n|_Y;\ \{f_n\} \subset Y,\ \lim_{n\to\infty} \| f_n - f \| = 0 \}.$$