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Non-commutative Algebraic Geometry

An Introduction

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F. VAN OYSTAEYEN A. VERSCHOREN

CONTENTS

Introduction.

It took us some time before we decided that this book is about "Non-commutative Algebraic Geometry". We realize that any non-commutative generalization of a commutative theory should be rooted in a thorough understanding of the heart of the commutative matter and, whereas the understanding of Algebraic Geometry is itself a non-trivial task, it becomes harder still if one sticks to the point of view that the obtained generalization should shed some new light upon the commutative theory.
Whether this book does or does not satisfy these requirements is subjected to the reader 's attitude towards Geometry. Subjectivity may be very unmathematical but it is perhaps allowable in the foregoing statement because of the extent of the topic as well as the fuzziness of its limits. Our point of view is that Algebraic Geometry describes the structure of geometric objects immersed in affine or projective space, utilizing ring- and sheaf-theoretic methods in exploiting the basic duality between these geometric objects and their morphisms and rings and ring homomorphisms. Therefore our first aim will be to obtain a non-commutative analogue of the Spec functor and the corresponding sheaf theory. However, functoriality of Spec with respect to ring homomorphisms is incompatible with the property that each ring should be recovered as the ring of global sections of the structure sheaf on Spec (R), unless one restricts to commutative rings. On the other hand, utilizing suitable localization in defining structure sheaves, it is not difficult to obtain a sheaf on Spec (R) such that R is the ring of global sections and such that Spec is functorial with respect to extensions of rings, i.e. ring homomorphisms $f:R\to S$, such that $f(R)$ and $Z_R(S) = \{s \in S;\ sr = rs \text{ for all } r \in f(R)\}$ generate the ring S. Then the fact that the stalks of this sheaf need not be local rings, presents another minor problem which can be solved by considering varietal spaces, which are ringed spaces endowed with three (!) interrelated structure sheaves. Now if, as in the commutative case, one wants non-commutative algebraic varieties to be determined by the set of closed points, if one hopes to have an analogue for

Hilbert's Nullstellensatz, if one is hankering after satisfactory theory of products of varieties and subvarieties, etc..., then, step by step, one is led to consider rings satisfying a polynomial identity (P.I. rings) affine over an algebraically closed field.

Now, the theory of rings satisfying polynomial identities has been flourishing the past decade and the prime ideal structure of these rings has been studied extensively, a.o. by C. Procesi, [136], [130], M. Artin [18] and M. Artin, W. Schelter [20], [21]. The only sheaves used in the so-called "geometry of P.I. rings" are sheaves of Azumaya algebras on certain open sets of the spectrum. Roughly speaking an Azumaya algebra can only appear as a stalk, when this stalk is strongly related to the center, hence precisely the non-Azumaya stalks, reflect the non-commutativity of the considered variety. Although a sheaf of Azumaya algebras on a particular dense set in the spectrum may already contain a lot of information, we feel that incorporating non-Azumaya stalks in our structure sheaves on varieties is an essential extra ingredient of the theory expounded here. In fact, an algebraic variety may usually be viewed as a covering of the underlying central variety, and the splitting of a point of the central variety into several points of the variety is reflected in the non-Azumaya-ness of certain stalks, i.e. the defect of certain localization being non-central ! The first part of the book, Chapter I to V, presents the necessary non-commutative algebra as well as the sheaf theoretical technicalities which will become the foundation for the geometrical theory.

We have kept the book as selfcontained as possible, although we have not strived for full generality everywhere. Well-known facts either in commutative Algebraic Geometry or in Ring Theory have been included as propositions without proofs, the exhaustive list of references makes it possible to trace any result used in this book. Note that we have included some results on P.I. rings, usually of a distinguished geometrical flavor, which have not been included in recent books. In the second part of the book, Chapters VI to X, we develop the "language" of algebraic k-varieties over an algebraically closed field. In particular, starting from affine k-varieties we construct the cellular k-varieties, these are algebraic k-varieties such that each point has an affine neighbourhood and these neighbourhoods have affine intersections. As it turns out, cellular varieties are likely to be the most fitting analogues of commutative algebraic varieties. In order to obtain non-commutative analogues of affine and projective space we have to introduce quasivarieties. These are very much the same as varieties but here we do not assume any Noetherian hypothesis. This generalization is being forced upon us because the rings for affine spaces turn out to be non-Noetherian in general. Utilizing the theory of graded rings established in Chapter III, we introduce projective space over an affine k-algebra. This allows to provide a more general framework and perhaps a more solid foundation for certain similar constructions carried out in some special cases by M. Artin in [20]. At this point it should be mentioned that the projective geometry has not been presented in full depth here; for example, the chapter on coherent sheaves over an algebraic k-variety is still rather unfinished, e.g. the properties of coherent sheaves over Proj (R) have scarcely been hinted at.

The definition of closed subvarieties presents no problem and most of their desired properties may be derived without much difficulties. On the other hand, it is impossible to define the categorical product of algebraic k-varieties, and therefore we have introduced the notion of a geometric product. Anyhow, this geometrical product suits our aims well enough and it turns out to be particularly effective in the study of cellular varieties. One should note that, the fact that "non-commutative algebraic groups" are Azumaya varieties, i.e. homeomorphic to the underlying central algebraic group, follows from some homogeneity argument and it has not been forced by the special nature of the geometric product used in the definition.

Chapter XI plays a special role in our set up, as a matter of fact it was only after the Riemann-Roch theorem for curves was established that the title of the book presented no more moral problems to us. The first section, that could

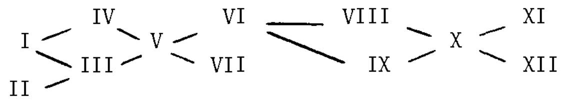
have been a separate chapter in his own right, deals with the study of k_o-rational points of a variety in case k_o is not necessarily algebraically closed. The second section uses birationality of varieties in reducing the study of curves "up to birationality" to the study of abstract curves which are given by a variety of maximal orders in a certain central simple algebra i.e. the function algebra of the variety. The latter function algebra represents, in the case of curves, an element of the Brauer group of a function field in one variable. Conversely elements of that Brauer group may be viewed as function algebras of certain non-commutative curves defined over the central curve (everything up to birationality) given by the function field in one variable. The version of the Riemann-Roch theorem we have included is based upon the theory of primes in central simple algebras, cf. J. Van Geel [170] , F. Van Oystaeyen [178] , which yields a suitable generalization of valuation theory of fields. Thus we define the genera of the elements of the Brauer group of an algebraic function field in one variable and express the classical relations between genus and dimensions of certain k-vector spaces of divisors. We obtain the canonical class by looking at the different of the central simple algebra and this is compatible with the results of E. Witt, [202] , but here we do obtain more information about the ring of constants.
In the final Chapter XII, we point out some other results, like M. Artin's version of the "Zariski main theorem", his use of proper and geometric morphisms and the relations between these concepts and integrality of extensions of P.I. rings. The topics in this chapter have not been worked out extensively, since we feel they have not yet reached a (semi-) final form, e.g. regularity of varieties seems to be linked to the theory of hereditary orders over regular rings but the latter is poorly developed as yet; we aim to come back to these problems later. The particular role of Chapter XI is once more highlighted in the problem of tracing back the relations between the set of genera of the elements of the Brauer group of a function field in one variable and the geometry of the commutative curve associated to it. With this we hope to have indicated

how the circle may be closed, now that some applications of the non-commutative theory in the study of commutative varieties seem to evolve.

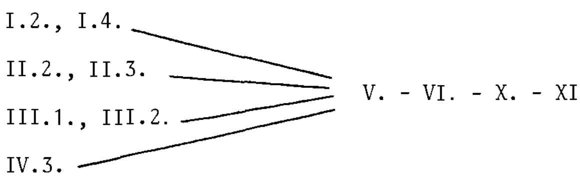
One thing about notation,. since the authors were having daily quarrels concerning the spelling of the term non-commutative (noncommutative) it will sometimes appear as noncommutative (non-commutative).

LEITFADEN

1.



2. The Shortest Way to the Riemann-Roch Theorem.



(You will have to take something for granted.)

I. GENERALITIES.

I.1. Simple Artinian Rings.

Throughout, R is an associative ring with unit. A left R-module M is said to be simple if M is non-zero and M has no submodules other than M and o. A direct sum of simple modules is called a semisimple module. It is well-known that $M \in R$-mod is semisimple if and only if every submodule of M is a direct summand in M. Clearly submodules and epimorphic images of semisimple modules are semisimple. A semisimple module is isotypic if it is a direct sum of isomorphic simple modules, the type of a simple module is given by its isomorphism class.
The socle of an arbitrary $M \in R$-mod is the sum of all simple submodules of M.

If $M \in R$-mod is semisimple then M is a right $\text{End}_R(M)$-module and M is again semisimple as such. Note also that the R-endomorphism ring of a semisimple $M \in R$-mod is the product of the $\text{End}_R(M_\alpha)$, where $M = \bigoplus_\alpha M_\alpha$ is the decomposition of M into isotypic components.

The ring R is semisimple if it is semisimple when considered as a left R-module.

I.1.1. Theorem. The following statements are equivalent :

a. R is semisimple.

b. Every $M \in R$-mod is semisimple.

c. Every $M \in R$-mod is projective.

d. Every $M \in R$-mod is injective.

Proof : Easy. For more detail cf. P. Cohn [39], or P. Ribenboim [145].

I.1.2. Schur's Lemma. If $M \in R$-mod is simple then $\text{End}_R(M)$ is a skew field.

This celebrated lemma applies to a celebrated theorem :

I.1.3. Wedderburn's Theorem. A semisimple ring is a direct sum of full matrix rings over skewfields and vice versa. The skew fields appearing are determined up to isomorphism by the structure of the ring and so is the size of the matrices.

A non-zero ring is said to be simple if it is semisimple and if it possesses no non-trivial ideals. By I.1.3. it is clear that a simple ring is Artinian and that every simple ring is a full matrix ring over a skew field and conversely Furthermore, the semisimple rings are just the finite direct products of simple rings (obviously they are also Artinian). In a way, the commutative version of all this may be summed up in "a non-commutative way", as follows : the center of a semisimple ring is a direct product of fields.

An $M \in R$-mod is faithful if the canonical homomorphism $R \to End_R(M)$ is injective. A ring R is primitive if there exists a simple faithful $M \in R$-mod (R is in fact "left" primitive). It is well-known that the primitive rings are exactly the dense rings of linear transformations of vector spaces over skew fields. Wedderburn's theorem yields that for a left Artinian ring, being simple is equivalent to being primitive. Furthermore, a ring is primitive if and only if it contains a maximal left ideal which contains no non-zero ideal.

The Jacobson radical J(R) of a ring R is defined to be the set of elements satisfying one of the equivalent properties listed in the following theorem.

I.1.4. Theorem. The following conditions for $r \in R$ are equivalent :

a. For each simple $M \in R$-mod, $rM = o$.

b. Each maximal left ideal of R contains r.

c. For all $x \in R$, 1-rx is left invertible in R.

a',b',c'. The right analogues of a,b,c.

The Jacobson radical is an ideal of R. If $J(R) = o$ then we say that R is semiprimitive and this is equivalent to the following statement: for each $r \neq o$ in R there is a simple $M \in R$-mod not annihilated by r i.e. $rM \neq o$. A useful application of these notions is :

I.1.5. Nakayama's Lemma. If $M \in R$-mod is finitely generated then $J(R)M = M$ implies $M = o$.

It is easily verified that the Jacobson radical is indeed a radical, viz. R/J(R) is semiprimitive for any ring R. Every left nilideal of R is contained in J(R). On the other hand J(R) need not be nilpotent, however we have :

<u>I.1.6. Proposition</u>. If R is left Artinian then J(R) is nilpotent. A left Artinian ring is semiprimitive if and only if the zero ideal is the unique nilpotent ideal.

<u>I.1.7. Example</u>. The ring $\begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ is right but not left Artinian. The ring k [[X]] of formal power series over the field k is a domain with non-zero Jacobson radical.

The Jacobson radical may be used to characterize left Artinian rings amongst left Noetherian rings :

<u>I.1.8. Proposition</u>. Let R be a left Noetherian ring, then R is left Artinian exactly then when R/J(R) is semisimple in R-mod and J(R) is nilpotent.

Since the center of a simple ring is a field, k say, it may be considered as a k-algebra. If a simple ring A has finite dimension over its center k then we say that A is a <u>k-central simple algebra</u> (or a <u>c.s.a.</u>). A criterion for checking whether a simple ring A is a c.s.a. will be provided in Section II.2, namely, a necessary and sufficient condition for this is that A satisfies some polynomial identities. We recall some almost-classical properties and theorems about c.s.a., without going into the theory of the Brauer group. For full detail on the subject we refer to [24],[49],[81],...

<u>I.1.9. Notation</u>. If R is a ring with center C then R^o denotes the <u>opposite ring</u> of R while the <u>enveloping ring</u> of R is $R^o \underset{C}{\otimes} R = R^e$.

<u>I.1.10. Proposition</u>. If A is a k-central simple algebra then $A^e \cong M_n(k)$ where $n = [A : k]$.

<u>I.1.11. Theorem</u>. (Azumaya-Nakayama). If A is k-central simple then for each k-algebra B there is a lattice-isomorphism between the ideals of B and the ideals of $A \underset{k}{\otimes} B$.

I.1.12. Corollaries. If A is k-central simple and B is a simple k-algebra then so is $A \underset{k}{\otimes} B$.

The class of k-central simple algebras is closed under taking tensor products.

If ℓ is a field extension of k then $A \underset{k}{\otimes} \ell$ is ℓ-central simple.

It is true that every derivation δ of a c.s.a. A is an inner derivation i.e. $\delta(a) = xa - ax$ for some $x \in A$.

For automorphisms fixing the center a similar statement is true; this is contained in :

I.1.13. Theorem (Skolem-Noether). Let R be a simple ring with center k, let S be any finite dimensional simple k-algebra. If ρ_1, ρ_2 are morphisms from S to R then for some unit u of R we have $\rho_2(s) = u\rho_1(s)u^{-1}$ for all $s \in S$.

I.1.14. Corollary. Automorphisms of finite-dimensional semisimple k-algebras which leave the center elementwise fixed are inner automorphisms.

Simple subalgebras of a c.s.a. are isomorphic if and only if they are conjugated.

The simple subalgebras of a c.s.a. are accurately described by the "double centralizer theorem". If B is a subring of A then $Z_A(B) = \{a \in A,\ ab = ba$ for all $b \in B\}$ is called the centralizer of B in A.

I.1.15. Theorem (R. Brauer). Let A be k-central simple and suppose that B is a simple subalgebra with center ℓ then $Z_A(B)$ is simple with center ℓ and $Z_A(Z_A(B)) = B$, $Z_A(\ell) = B \underset{k}{\otimes} Z_A(B)$. The k-dimensions of A,B and $Z_A(B)$ are related by : $[A : k] = [B : k] . [Z_A(B) : k]$ and if $[B : k] = n$ is finite then : $A \underset{k}{\otimes} B^\circ \cong M_n(Z_A(B))$.

I.1.16. Corollary. Let ℓ be a field extension of k contained in a k-central simple algebra A then the following statements are equivalent :

a. $Z_A(\ell) = \ell$.

b. $[A : k] = [\ell : k]^2 = [A : \ell]^2$

c. The field ℓ is maximal amongst the commutative subrings of A.

A splitting field of a k-central simple algebra A is an extension field of k, say ℓ, such that $A \underset{k}{\otimes} \ell \cong M_n(\ell)$. A maximal commutative subfield of A is a splitting field. Any splitting field contains a finite dimensional splitting field. For the theory of the Brauer group one may consult [81] or [168]. The theory of generic abelian crossed products, cf. Van Oystaeyen [171] and the generalization by Amitsur, Saltman [10], may be brought to bear on Sections II.3. and III2, but it would take us too far to include it here.
To end this introductory part let us mention :

I.1.17. Theorem.(Cartan-Hua-Brauer). Let A be a k-central simple algebra and let B be a subalgebra. If B is globally invariant under inner automorphisms of A then either $B \subset k$ or $B = A$.

The previous survey should make it clear that simple algebras, in particular c.s.a., have been extensively studied and that their structure is reasonably well-known. We have omitted the implications of the theory for the theory of representation of finite groups, cf. [49], [39], although this may be considered to be a main motivation for "embeddings" of non-commutative rings into simple rings and into central simple algebras.
The first problem is being treated in A. Goldie's theorems while the central simple algebra-case is the topic of Section II.2. i.e. Posner's theorem for P.I. rings. A major problem in the non-commutative theory is that the formation of total rings of fractions of certain integral domains need not be possible. Let us consider the following integral domains for which rings of fractions may still be constructed.

An integral domain R is said to be a left Ore domain if the intersection of any two non-zero left ideals is non-zero. Right Ore domains are defined in a similar way. There do exist integral domains which are neither a left or a right Ore domain; indeed, the free algebra generated by two symbols over a field has that property. Even better, (or worse?) there exist right Ore domains which are not left Ore domains e.g. the following ring of twisted polynomials. Let ℓ

be any field, σ an isomorphism of ℓ onto a proper subfield of ℓ, and consider the ring, $\ell[x,\sigma]$, of "polynomials" in the indeterminate x with multiplication defined by the rule : $rx^n = x^n\sigma^n(r)$ for $r \in \ell$. It is not too hard to verify that $\ell[x,\sigma]$ is a right Ore domain but not a left Ore domain.

I.1.18. Theorem. If R is an integral domain then the following statements are equivalent :

a. R is a left Ore domain.
b. There is a skew field containing R as a subring, S say, such that $S = \{b^{-1}a;\ a,b \in R,\ b \neq o\}$.
c. R has finite Goldie dimension in R-mod, i.e. R cannot contain an infinite direct sum of non-zero submodules.

We now try to extend this procedure to rings which are not necessarily integral domains.

A regular element in a ring R is an element r such that $rx \neq o$ and $xr \neq o$ for all non-zero $x \in R$. A classical left quotient ring for R is a ring Q containing R as a subring such that regular elements of R are units in Q and $Q = \{b^{-1}a;\ a,b \in R,$ b regular$\}$.

Let R be any ring and let S be a multiplicatively closed subset of R, then R is said to satisfy the left Ore condition with respect to S if, for every $s \in S$, $r \in R$ there exist $s' \in S$, $r' \in R$ such that $s'r = r's$. In a rather straightforward way one verifies :

I.1.19. Theorem. The ring R has a left quotient ring at S i.e. there exists a ring R_S and a ring homomorphism $j : R \to R_S$ such that $j(s)$ is invertible in R_S for each $s \in S$, every element of R_S is of the form $j(s)^{-1}\, j(r)$ with $s \in S$ and $j(r) = o$ if and only if $sr = o$ for some $s \in S$, exactly when R satisfies the left Ore condition with respect to s and S is left reversible ($rs = o$ with $s \in S$ implies $s'r = o$ for some $s' \in S$).

In particular, if S is the set of regular elements then the left Ore condition with respect to S is just called the left Ore condition. In this case S is clearly reversible and one easily derives from I.1.19. that the left Ore condition implies that R has a classical left quotient ring. Now Goldie's theorems provide a criterion for checking whether such a classical left quotient ring is semisimple or simple.

The left annihilator of an element $m \in M$, $M \in R$-mod, is the left ideal $\{r \in R, rm = o\}$. A left Goldie-ring R is a ring which has finite Goldie-dimension and such that left annihilators satisfy the ascending chain condition. An essential submodule N of $M \in R$-mod is a left R-module such that $N \subset M$ intersects all non-zero submodules of M non-trivially. An essential left ideal is then just an essential left submodule of R. The singular submodule of any $M \in R$-mod, denoted by $t_s(M)$, is defined to be the set $\{x \in M, Jx = o$ for some essential left ideal J of R$\}$. We define $t_s^{\ell}(R)$, $t_s^r(R)$ correspondingly; ℓ and r refer to left or right corresponding to whether R is considered as a left or a right R-module. Clearly, both $t_s^{\ell}(R)$ and $t_s^r(R)$ are ideals of R.

I.1.20. Lemma. If R satisfies the ascending chain condition on left annihilators, then $t_s^{\ell}(R)$ is a nilpotent ideal of R.

I.1.21. Corollary. A semiprime ring R is a left Goldie ring if and only if $t_s^{\ell}(R) = o$ and R has finite Goldie dimension.

I.1.22. Lemma. Let R be a ring with finite Goldie dimension such that $t_s^{\ell}(R) = o$. An $r \in R$ is regular if and only if its left annihilator is zero, and also if and only if Rr is an essential left ideal.

I.1.23. Proposition. In a semiprime left Goldie ring R the essential left ideals are exactly the left ideals of R which contain a regular element.

I.1.24. Theorem. (Goldie's second Theorem). The following conditions are equivalent :

a. R is a semiprime left Goldie ring.

b. R allows a classical left quotient ring which is semisimple.

I.1.25. Corollary. (Goldie's First Theorem). R allows a simple left quotient ring if and only if it is a prime left Goldie ring.

I.1.26. Proposition. If R is a semiprime left and right Goldie ring then R allows a semisimple ring Q for a classical left and right quotient ring. If an arbitrary ring R has a classical left quotient ring and also a classical right quotient ring, then these coincide.

I.1.27. Example. Let A be a k-central simple algebra and let C be a subring of k such that k is the field of fractions of C; let R be a C-algebra contained in A and containing a k-basis of A. Then R is a prime left and right Goldie ring (see also Section I.2.).

I.1.28. Comment. 1. Necessary and sufficient conditions for R to have a c.s.a. as left (and right) quotient ring are in II.2.

2. More general techniques of localization will be necessary in the sheaf theory of Chapter III, for these we refer to Sections I.3, I.4 and IV.

I.1.19. References. For the Jacobson radical and parts of the theory developed from and about it we refer to N. Jacobson's book [87]. Much of the basic theory of c.s.a. is of course in A. Albert's book [1], and also in B. Van der Waerden's [163]. Very nice compilations of the material we surveyed are I. Herstein [81] and P. Cohn [39]. A recent treatment of Goldie's theorems is given by K. Goodearl in [72]; of course one also may consult Goldie's papers [66],[68]. An easy example of a right but not left Goldie ring is a ring of twisted polynomials with respect to an injective (not bijective) endomorphism, more general results about these may be found in G. Cauchon's thesis, [33] .