MATRICES AND TRANSFORMATIONS

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To Steven, Paul,
and Donna

Preface

MATRIX ALGEBRA is a significant topic in contemporary mathematics curricula. This book presents the fundamental concepts of matrix algebra, first in an intuitive framework and then in a more formal manner. A variety of interpretations and applications of the elements and operations considered are included. In particular, the use of matrices in the study of transformations of the plane is stressed. The purpose of this book is to familiarize the reader with the role of matrices in abstract algebraic systems, and to illustrate its effective use as a mathematical tool in geometry.

Chapters 1 and 2 include those basic concepts of matrix algebra that are important in the study of physics, statistics, economics, engineering, and mathematics. Matrices are considered as elements of algebra. The concept of a linear transformation of the plane and the use of matrices in discussing such transformations is illustrated in Chapter 3. Some aspects of the algebra of transformations and its relation to the algebra of matrices are included here. Chapter 4 contains material usually not found in an introductory treatment of matrix algebra, including an application of the properties of eigenvalues and eigenvectors to the study of the conics.

The motivations of the concepts presented are included wherever appropriate. Considerable attention has been paid to the formulation of precise definitions and statements of theorems. The proofs of most of the theorems are included in detail in this book.

Many illustrative examples have been included to facilitate the use of the book both for individual study and for summer institutes for teachers of mathematics. This book contains enough material for a one-semester course at the college level or for enrichment programs at the high school level. Also, it may be used together with my book, *Vectors and Their Applications*, for a course in linear algebra. There are a sufficient number of exercises, which range from routine computations to proofs of theorems that extend the theory of the subject. Answers are provided to the odd-numbered exercises.

It is assumed that the reader has some understanding of the basic fundamentals of vector algebra. All of Chapters 1 and 2 may be studied, however, without any previous knowledge of vectors. Certain derivations and interpretations of material contained in Chapters 3 and 4 may be omitted by the reader who is unfamiliar with geometric vectors.

To the many students and teachers who have contributed to my understanding of matrix theory and to the preparation of this book, I acknowledge a debt of gratitude. I am particularly grateful to Dr. Bruce E. Meserve of the University of Vermont for suggesting that this book be written and for his invaluable comments and criticisms throughout its development. I wish to express my appreciation to my wife, Betty, for typing the final manuscript as well as the preliminary versions. A special note of appreciation is due the editorial-production staff of Prentice-Hall, Inc., for their cooperation in the production of this book.

Anthony J. Pettofrezzo

Contents

chapter 1

Matrices

1-1 Definitions and Elementary Properties

In many branches of the physical, biological, and social sciences it is necessary for scientists to express and use a set of numbers in a rectangular array. Indeed, in many everyday activities it is convenient, if not necessary, to use sets of numbers arranged in rows and columns for keeping records, for purposes of comparison, and for a variety of other reasons.

Consider a company that manufactures three models of typewriters: an electric model, a standard model, and a portable model. If the company wishes to compare the units of raw material and labor involved in one month's production of each of these models, an *array* may be used to present the data:

$$\begin{array}{l} \\ \\ \textit{Units of material} \\ \textit{Units of labor} \end{array} \begin{array}{ccc} \textit{Electric} & \textit{Standard} & \textit{Portable} \\ \textit{model} & \textit{model} & \textit{model} \\ \end{array}$$
$$\begin{array}{l} \textit{Units of material} \\ \textit{Units of labor} \end{array} \begin{pmatrix} 20 & 17 & 12 \\ 6 & 8 & 5 \end{pmatrix}.$$

The units used are not intended to be realistic but merely to illustrate an oversimplified application of an array of real numbers. Units of material for the three models comprise the first row of the array, units of labor the second row, and units of production for each model (material and labor) the columns of the array. If the pattern in which the units are to be recorded is clearly defined in advance, this rectangular array may be presented simply as:

$$\begin{pmatrix} 20 & 17 & 12 \\ 6 & 8 & 5 \end{pmatrix}. \tag{1-1}$$

A second example of the use of rectangular arrays of real numbers is one that might be used by a basketball coach who wishes to keep a record of the scoring performances of three of his players. Consider the following array:

$$\begin{array}{l} \\ \textit{Player A} \\ \textit{Player B} \\ \textit{Player C} \end{array} \begin{array}{ccc} \textit{Games} & \textit{Field goals} & \textit{Free throws} \\ \end{array}$$
$$\begin{array}{l} \textit{Player A} \\ \textit{Player B} \\ \textit{Player C} \end{array} \begin{pmatrix} 16 & 110 & 62 \\ 14 & 85 & 42 \\ 16 & 73 & 55 \end{pmatrix}$$

or simply

$$\begin{pmatrix} 16 & 110 & 62 \\ 14 & 85 & 42 \\ 16 & 73 & 55 \end{pmatrix}. \tag{1-2}$$

Rectangular arrays of elements a_{ij} such as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{1-3}$$

are called **matrices** (singular: **matrix**). Each element a_{ij} has two indices: the **row index**, i, and **column index**, j. The elements $a_{i1}, a_{i2}, \ldots, a_{in}$ are the elements of the ith row, and the elements $a_{1j}, a_{2j}, \ldots, a_{mj}$ are the elements of the jth column. The element a_{ij} is the element contained simultaneously in the ith row and jth column. For example, the element a_{21} of matrix (1-1) is equal to 6; that is, a_{21} is the element in the second row and first column.

A matrix of m rows and n columns is called a matrix of **order** m by n. Thus, matrix (1-1) is of order 2 by 3, while matrix (1-2) is of order 3 by 3. In general, when the number of rows equals the number of columns, the matrix is called a **square matrix.** A square matrix of order n by n is said, simply, to be of order n. Matrix (1-2) is an example of a square matrix of order three.

If each of the elements of a matrix is a real number, the matrix is called a **real matrix.** Unless otherwise stated, we shall be concerned only with real matrices.

Whenever it is convenient, matrices will be denoted symbolically by capital letters $A, B, C, \ldots$, or by $((a_{ij})), ((b_{ij})), ((c_{ij})), \ldots$ where $a_{ij}, b_{ij}, c_{ij}, \ldots$, respectively, represent the general elements of the matrices.

Example 1 Construct a square matrix $((a_{ij}))$ of order three where $a_{ij} = 3i - j^2$.

If $a_{ij} = 3i - j^2$, then

$$a_{11} = 3(1) - (1)^2 = 2,\ a_{12} = 3(1) - (2)^2 = -1,\ a_{13} = 3(1) - (3)^2 = -6,$$
$$a_{21} = 3(2) - (1)^2 = 5,\ a_{22} = 3(2) - (2)^2 = 2,\ a_{23} = 3(2) - (3)^2 = -3,$$
$$a_{31} = 3(3) - (1)^2 = 8,\ a_{32} = 3(3) - (2)^2 = 5,\ a_{33} = 3(3) - (3)^2 = 0.$$

Hence, the desired matrix is

$$\begin{pmatrix} 2 & -1 & -6 \\ 5 & 2 & -3 \\ 8 & 5 & 0 \end{pmatrix}.$$

Two matrices $((a_{ij}))$ and $((b_{ij}))$ are said to be **equal** if and only if they are of the same order, and $a_{ij} = b_{ij}$ for all pairs (i, j).

Example 2 Determine whether or not the matrices of each pair are equal:

(a) $\begin{pmatrix} 4 & 1 & 2 \\ -2 & 3 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$; **(b)** $\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$;

(c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$; **(d)** $\begin{pmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The matrices in **(a)** cannot be equal since they are not of the same order. The matrices in **(b)** are equal. The matrices in **(c)** are equal if and only if $a = b = c = 0$. Although the matrices in **(d)** are of the same order, they are not equal, since not all corresponding elements are equal.

Consider again matrix (1-2) which represents the scoring performances of three basketball players for one season. Suppose that the matrix representing the scoring performances of these players in the next season of play is

$$\begin{pmatrix} 18 & 142 & 98 \\ 18 & 83 & 33 \\ 15 & 103 & 60 \end{pmatrix}.$$

A matrix representing the combined scoring performance of each of the three players during two seasons may be obtained by adding the corresponding entries of the two matrices:

$$\begin{pmatrix} 16 & 110 & 62 \\ 14 & 85 & 42 \\ 16 & 73 & 55 \end{pmatrix} + \begin{pmatrix} 18 & 142 & 98 \\ 18 & 83 & 30 \\ 15 & 103 & 60 \end{pmatrix} = \begin{pmatrix} 34 & 252 & 160 \\ 32 & 168 & 72 \\ 31 & 176 & 115 \end{pmatrix}.$$

That is, in two seasons players A, B, and C participated in 34, 32, and 31 games, scored 252, 168, and 176 field goals, and 160, 72, and 115 free throws, respectively.

In general, the addition of two matrices $((a_{ij}))$ and $(b_{ij}))$ is defined if and only if the matrices are of the same order. If $((a_{ij}))$ and $((b_{ij}))$ are matrices of the same order, then the **sum** $((a_{ij})) + ((b_{ij}))$ is defined as a third matrix $((c_{ij}))$ of that same order where each element c_{ij} satisfies the condition $c_{ij} = a_{ij} + b_{ij}$.

Consider any three real matrices of order m by n: $A = ((a_{ij}))$, $B = ((b_{ij}))$, and $C = ((c_{ij}))$. Since the addition of real numbers is commutative, $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for all pairs (i, j) and

$$A + B = B + A. \tag{1-4}$$

Since the addition of real numbers is associative, $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$ for all pairs (i, j) and

$$(A + B) + C = A + (B + C). \tag{1-5}$$

Thus, *the addition of real matrices is commutative and associative.*

A **null matrix** or **zero matrix,** denoted by 0, is a matrix wherein all of the elements are zero. For every matrix A of order m by n there exists a zero matrix of order m by n such that $A + 0 = 0 + A = A$. This zero matrix of order m by n is the *additive identity element* for the set of all matrices of order m by n.

Example 3 Find the sum of matrices A and B where

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -1 & 3 \\ 2 & 0 & 5 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 2+3 & 1+(-1) & 1+3 \\ 0+2 & -3+0 & 4+5 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 4 \\ 2 & -3 & 9 \end{pmatrix}.$$

Consider again matrix (1-1) which represents the units of material and labor involved in one month's production of three models of typewriters. Suppose the manufacturing company wishes to double its production of each model. Then the matrix

$$\begin{pmatrix} 40 & 34 & 24 \\ 12 & 16 & 10 \end{pmatrix}$$

would represent the units of material and labor involved in a single month's production of the three models of typewriters. It is convenient to represent the doubling of the entries in matrix (1-1) as a product of the matrix and the real number 2; that is,

$$2\begin{pmatrix} 20 & 17 & 12 \\ 6 & 8 & 5 \end{pmatrix} = \begin{pmatrix} 40 & 34 & 24 \\ 12 & 16 & 10 \end{pmatrix}.$$

Note that

$$2\begin{pmatrix} 20 & 17 & 12 \\ 6 & 8 & 5 \end{pmatrix} = \begin{pmatrix} 20 & 17 & 12 \\ 6 & 8 & 5 \end{pmatrix} + \begin{pmatrix} 20 & 17 & 12 \\ 6 & 8 & 5 \end{pmatrix}.$$

In general, the product of a real number (scalar) k and a matrix $((a_{ij}))$, denoted by $k((a_{ij}))$ or by $((a_{ij}))k$, is called the scalar multiple of the matrix $((a_{ij}))$ by k. The **scalar multiple** $k((a_{ij}))$ is defined as a matrix wherein the elements are products of k and the corresponding elements of $((a_{ij}))$. Since the multiplication of real numbers is commutative,

$$k((a_{ij})) = ((a_{ij}))k = ((ka_{ij})). \tag{1-6}$$

Notice that for any real number k and any real matrix $A = ((a_{ij}))$, the matrices A and kA are of the same order. In particular, if $k = -1$, then

$$A + (-1)A = ((a_{ij})) + ((-a_{ij})) = 0.$$

Thus, $(-1)A$ is the *additive inverse* of A. If B and A are any two matrices of the same order, the **difference** $B - A$ is defined by the relation

$$B - A = B + (-1)A. \tag{1-7}$$

In general, the scalars may be considered as scalar coefficients, and any algebraic sum of scalar multiples of matrices of the same order satisfies certain laws. For example, if k and l are scalars and A and B are matrices of the same order, then

$$kA + lA = (k + l)A, \tag{1-8}$$

$$klA = k(lA) = l(kA) = (kl)A, \tag{1-9}$$

and

$$k(A + B) = kA + kB. \tag{1-10}$$

Furthermore, if $kA = 0$, then either $k = 0$ or A is a zero matrix.

Example 4 Find $3A - 2B$ where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix}.$$

$$3A - 2B = 3\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix} = 3\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 3(1) & 3(2) \\ 3(3) & 3(0) \end{pmatrix} + \begin{pmatrix} -2(1) & -2(3) \\ -2(0) & -2(-4) \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 6 \\ 9 & 0 \end{pmatrix} + \begin{pmatrix} -2 & -6 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 9 & 8 \end{pmatrix}.$$

Exercises

1. Construct a square matrix $((a_{ij}))$ of order three where $a_{ij} = i^2 + 2j - 3$.
2. Construct a matrix $((a_{ij}))$ of order 3 by 2 where $a_{ij} = i^2 - ij$.
3. In the square matrix $((a_{ij}))$ of order two describe the position of the elements for which **(a)** $i = 2$, **(b)** $j = 1$, and **(c)** $i = j$.

4. If

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 & 0 & 3 \\ 1 & -4 & -1 \end{pmatrix},$$

then find **(a)** $A + B$; **(b)** $A - B$; **(c)** $A + 3B$.

5. Verify the associative law of addition of matrices (1-5) for

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -2 & 5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 2 & 7 & -1 \\ -2 & 1 & 3 \end{pmatrix}.$$

6. Find the additive inverse of the matrix

(a) $\begin{pmatrix} 2 & 5 & 3 \\ 0 & 2 & -1 \end{pmatrix}$; **(b)** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

7. Solve the matrix equation

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 1 & 0 \end{pmatrix}.$$

8. Prove that the set of real matrices of order m by n forms a commutative group under matrix addition.

1-2 Matrix Multiplication

Consider a system of linear equations such as

$$\begin{cases} 2x - y + 2z = 1 \\ x + 2y - 4z = 3 \\ 3x - y + z = 0. \end{cases} \tag{1-11}$$

This system may be represented by a single matrix equation:

$$\begin{pmatrix} 2x - y + 2z \\ x + 2y - 4z \\ 3x - y + z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}. \tag{1-12}$$

The coefficients of x, y, and z may be obtained either from (1-11) or (1-12). In both cases the solution depends upon these coefficients. The **matrix of coefficients** is

$$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -4 \\ 3 & -1 & 1 \end{pmatrix}.$$

The coefficients of each variable are positioned in a column, and coefficients of the variables of each equation are located on a row. It is customary and convenient to think of this matrix of coefficients as an operator that acts upon a column matrix of the variables:

$$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -4 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y + 2z \\ x + 2y - 4z \\ 3x - y + z \end{pmatrix}. \tag{1-13}$$

Then, the system of equations (1-11) may be represented by the single matrix equation

$$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -4 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}. \tag{1-14}$$

Use of the matrix of coefficients as an operator in (1-13) requires the introduction of *matrix multiplication.* Notice that the element $2x - y + 2z$ may be obtained from the matrices

$$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -4 \\ 3 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

by summing the products of the elements of row one of the matrix of coefficients and the corresponding elements of the column matrix of variables, taken in order; that is,

$$(2)(x) + (-1)(y) + (2)(z) = 2x - y + 2z.$$

Similarly, the element $x + 2y - 4z$ may be obtained by summing the products of the elements of row two of the matrix of coefficients and the corresponding elements of the column matrix of variables, taken in order; that is,

$$(1)(x) + (2)(y) + (-4)(z) = x + 2y - 4z.$$

In a similar manner, using the elements of row three of the matrix of coefficients, we obtain

$$(3)(x) + (-1)(y) + (1)(z) = 3x - y + z.$$

In general, the **product** AB of two matrices A and B is defined to be a matrix C such that the element in the ith row and jth column of C is obtained by summing the products of the elements of the ith row of A and the corresponding elements of the jth column of B, taken in order. Notice that the number of columns of A must be the same as the number of rows of B. If $A = ((a_{ij}))$ is a matrix of order m by n and $B = ((b_{ij}))$ is a matrix of order n by r, then $C = ((c_{ij}))$ is a matrix of order m by r where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}. \tag{1-15}$$

We may use summation notation and write (1-15) as

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}. \tag{1-16}$$

When the number of columns of a matrix A is equal to the number of rows of a matrix B, the product AB exists, and the matrices A and B are said to be **conformable** for the product AB. Two matrices can be multiplied only when they are conformable. In the product AB, B is sometimes spoken of as being **premultiplied** by A, and A as being **postmultiplied** by B. Even if the product AB exists, the product BA may not exist since matrices A and B may not be conformable for product BA. This illustrates an important property of matrix multiplication, namely, that it is, in general, not commutative.

Example 1 Find the products AB and BA, if they exist, where

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix}.$$

Matrices A and B are conformable for the product AB since the number of columns of A is equal to the number of rows of B. Hence, AB exists. Furthermore, AB is of order 2 by 3 since A is of order 2 by 2 and B is of order 2 by 3. By definition, the general element of AB is given as $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j}$. Then

$$c_{11}=(2)(3)+(3)(1), \quad c_{12}=(2)(-2)+(3)(0), \quad c_{13}=(2)(2)+(3)(-1),$$
$$c_{21}=(1)(3)+(-4)(1), \quad c_{22}=(1)(-2)+(-4)(0), \quad c_{23}=(1)(2)+(-4)(-1);$$

that is,

$$c_{11} = 9, \quad c_{12} = -4, \quad c_{13} = 1,$$
$$c_{21} = -1, \quad c_{22} = -2, \quad c_{23} = 6.$$

Therefore,

$$AB = \begin{pmatrix} 9 & -4 & 1 \\ -1 & -2 & 6 \end{pmatrix}.$$

The product BA does not exist since the matrices B and A are not conformable for the product BA.

If the elements of any row or column of a matrix are considered to represent the components of a vector; for every pair of values (i, j), the element c_{ij} of (1-15) is the scalar (sometimes called the dot or inner) product of the ith row vector of A and the jth column vector of B. A matrix consisting of a single row is sometimes called a **row matrix** or a **row vector**; a matrix consisting of a single column is sometimes called a **column matrix** or a **column vector.**

Example 2 Find the matrix products AB and BA of the row vector $A = (1 \quad 2 \quad 3)$ and the column vector

$$B = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}.$$

Since A is of order 1 by 3 and B is of order 3 by 1, the matrices are conformable regardless of the order in which they are considered. Hence, the products AB and BA both exist:

$$AB = ((1)(-2) + (2)(4) + (3)(1)) = (9);$$

$$BA = \begin{pmatrix} (-2)(1) & (-2)(2) & (-2)(3) \\ (4)(1) & (4)(2) & (4)(3) \\ (1)(1) & (1)(2) & (1)(3) \end{pmatrix} = \begin{pmatrix} -2 & -4 & -6 \\ 4 & 8 & 12 \\ 1 & 2 & 3 \end{pmatrix}.$$

Note that the product AB may be considered a matrix whose only element represents the scalar product of two vectors whose components are the elements of A and B, respectively.

Example 3 Prove that $C(A + B) = CA + CB$ where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix}.$$

$$C(A + B) = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \left[\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ 21 & 13 \\ 6 & 0 \end{pmatrix};$$

$$CA + CB = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 4 \\ 10 & 2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} -2 & -10 \\ 11 & 11 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ 21 & 13 \\ 6 & 0 \end{pmatrix};$$

hence, $C(A + B) = CA + CB$.

Example 3 is an illustration of a general theorem of matrix algebra.

Theorem 1-1 *The multiplication of matrices is distributive with respect to addition.*

Proof: Let $A = ((a_{ij}))$ and $B = ((b_{ij}))$ be matrices of order m by n, and let $C = ((c_{ij}))$ be a matrix of order k by m. Then $A + B$, CA, CB, and $C(A + B)$ exist. The elements of the ith row of C are

$$c_{i1}, c_{i2}, \ldots, c_{im},$$

and the elements of the jth column of $A + B$ are

$$a_{1j} + b_{1j}, a_{2j} + b_{2j}, \ldots, a_{mj} + b_{mj}.$$

Therefore, the ijth element of $C(A + B)$ is

$$c_{i1}(a_{1j} + b_{1j}) + c_{i2}(a_{2j} + b_{2j}) + \cdots + c_{im}(a_{mj} + b_{mj});$$

that is,

$$(c_{i1}a_{1j} + c_{i2}a_{2j} + \cdots + c_{im}a_{mj}) + (c_{i1}b_{1j} + c_{i2}b_{2j} + \cdots + c_{im}b_{mj}),$$

the sum of the ijth elements of CA and CB, respectively. Hence,

$$C(A + B) = CA + CB. \tag{1-17}$$

Equation (1-17) represents the *left-hand distributive property* of matrix multiplication. The *right-hand distributive property*

$$(A + B)C = AC + BC \tag{1-18}$$

also is valid provided $A + B$, AC, BC, and $(A + B)C$ exist. Note that $C(A + B)$ and $(A + B)C$ generally are not equal.

Example 4 Prove that $A(BC) = (AB)C$ where

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

$$A(BC) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ 16 & 2 \end{pmatrix};$$

$$(AB)C = \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 & -1 \\ 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ 16 & 2 \end{pmatrix};$$

hence, $A(BC) = (AB)C$.

Example 4 is an illustration of one of the most important theorems of matrix algebra.

Theorem 1-2 *The multiplication of matrices is associative.*

Proof: Let $A = ((a_{ij}))$, $B = ((b_{ij}))$, and $C = ((c_{ij}))$ be matrices of order k by m, m by n, and n by p, respectively. Then the products AB, BC, $A(BC)$, and $(AB)C$ exist. The elements of the ith row of A are a_{i1}, $a_{i2}, \ldots, a_{im}$, and the elements of the jth column of BC are $b_{11}c_{1j} + b_{12}c_{2j}$

$+ \cdots + b_{1n}c_{nj}$, $b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2n}c_{nj}, \ldots, b_{m1}c_{1j} + b_{m2}c_{2j} + \cdots + b_{mn}c_{nj}$. Therefore, the ijth element of $A(BC)$ is

$$\begin{aligned} &a_{i1}(b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1n}c_{nj}) + \\ &a_{i2}(b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2n}c_{nj}) + \\ &\cdots + a_{im}(b_{m1}c_{1j} + b_{m2}c_{2j} + \cdots + b_{mn}c_{nj}); \end{aligned}$$

that is,

$$\begin{aligned} &(a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{im}b_{m1})c_{1j} + \\ &(a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{im}b_{m2})c_{2j} + \\ &\cdots + (a_{i1}b_{1n} + a_{i2}b_{2n} + \cdots + a_{im}b_{mn})c_{nj}, \end{aligned}$$

which represents the ijth element of $(AB)C$. Hence,

$$A(BC) = (AB)C. \qquad \text{(1-19)}$$

Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note that $AB = 0$, but neither A nor B is a zero matrix. In matrix algebra it is possible to have **zero divisors.** These are nonzero elements whose product is zero. In the algebra of real numbers it can be shown that zero divisors do not exist; that is, if a and b are real numbers and $ab = 0$, then $a = 0$ or $b = 0$.

Exercises

1. Find AB and BA, if they exist, where

$$A = \begin{pmatrix} 3 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 6 & -1 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 4 \end{pmatrix}.$$

2. Use the matrices given to verify that the multiplication of square matrices generally is not commutative:

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}.$$

3. Verify the associative property of matrix multiplication (1-19) for

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

4. Verify the right-hand distributive property (1-18) for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 3 & -1 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 2 \end{pmatrix}.$$

5. Find:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

6. Determine conditions on m, n, and p such that the product AB of matrix A of order m by n and matrix B of order n by p is a square matrix.

7. Given

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix},$$

show that $(A + B)(A - B) \neq A^2 - B^2$.

8. Determine a necessary and sufficient condition for (a) $(A + B)(A + B) = A^2 + B^2$; (b) $(A + B)(A - B) = A^2 - B^2$.

9. Find all matrices A such that

(a) $\begin{pmatrix} 3 & 0 \\ 5 & 0 \end{pmatrix} A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; (b) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} A = A \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$.

10. Prove that $AB = BA$ where

$$A = \begin{pmatrix} r & s \\ -s & r \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} m & n \\ -n & m \end{pmatrix}.$$

11. Find A^3 if $A^3 = A\,(AA)$ and

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}.$$

12. If

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

discuss the nature of A^n where n is a positive integer. (Note: The sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, . . . , where each term is the sum of the two preceding terms, is called a *Fibonacci sequence*.)

13. If $AC = CA$ and $BC = CB$, prove that $C(AB + BA) = (AB + BA)C$.

14. Prove that the set of *Pauli matrices*,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$E = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad F = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad G = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

forms a group under matrix multiplication. The Pauli matrices are used in the study of atomic physics.