EDITORS

A. AIZPURU-TOMÁS
F. LEÓN-SAAVEDRA

PROCEEDINGS OF THE FIRST INTERNATIONAL SCHOOL

ADVANCED COURSES OF MATHEMATICAL ANALYSIS I

ADVANCED COURSES OF MATHEMATICAL ANALYSIS I

This page intentionally left blank

EDITORS

A. AIZPURU-TOMÁS

F. LEÓN-SAAVEDRA

Universidad de Cádiz, Spain

PROCEEDINGS OF THE FIRST INTERNATIONAL SCHOOL

ADVANCED COURSES OF MATHEMATICAL ANALYSIS I

Cádiz, Spain 22–27 September 2002

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401–402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

ADVANCED COURSES OF MATHEMATICAL ANALYSIS I
Proceedings of the First International School

ISBN 981-256-060-2

Printed in Singapore by World Scientific Printers (S) Pte Ltd

Preface

An idea, a dream and then a reality. This is the manner we can describe the gestation of the First International Course of Mathematical Analysis in Andalucia.

Our small research group, working in the University of Cadiz, considers that, nowadays, Andalucia has a remarkable scientific and cultural importance as well as an indisputable international presence and this idea should take shape in some type of periodic event.

Our main aims were:

(i) That the different research groups in Andalucia working on mathematical analysis should meet each other and collaborate.

(ii) That the young researchers working in these groups could have access to the most advanced research lines.

Besides, we consider it of great importance to unite efforts in order to guarantee a solid education in our young researchers.

The Course was held in 2002, from September 23 to 27, in the historical part of Cadiz, a beautiful city surrounded by the Atlantic Ocean.

This book is the first volume in a series of advanced courses of Mathematical Analysis. The authors of this collection are recognized experts with an extensive research and educational experience. The authors of the first volume are: Yoav Benyamini, Manuel González Ortiz, Vladimír Müller, Simeón Reich (co-authored with E. Matouskova and A. J. Zaslavski) and Ángel Rodríguez Palacios.

The article by Benyamini is the only updated survey of the exciting and active area of the classification of Banach spaces under uniformly continuous maps.

The article by González is a pioneer introduction to the theory of local duality for Banach Spaces.

The paper, Genericity in nonexpansive mapping theory, by Eva Matoušková, Simeón Reich and Alexander J. Zaslavski, provides an up-to-date detailed overview of the applications of the generic method to nonexpansive mapping theory.

The article by Vladimír Müller provides a survey of results and ideas concerning orbits of operators and related notions of weak and polynomial orbits. The Scott Brown technique to obtain invariant subspaces is carefully exposed.

The survey paper by Rodríguez Palacios collects the results on absolute-valued algebras since the pioneering works of Ostrwski, Mazur, Albert, and Wright to the more recent developments. The celebrated Urbanik–Wright paper on the topic is fully reviewed. The survey contains in addition some new results, and several new proofs of known results. As a matter of fact, it will be the authoritative reference for the optimal version of results scattered in the literature through many separate papers.

We want to express our gratitude to all of them. We also want to thank Professors Tomás Domínguez Benavides and Ángel Rodríguez Palacios for being the first in supporting the idea of celebrating the Course.

We thank from the heart every participant for their presence: the mature researchers as well as the young ones still in their educational period. Of all of them we keep pleasant memories.

Besides, we want to thank Professor María Victoría Velasco for assuming the responsibility of organizing the second course in September 2004 in the gorgeous city of Granada.

Finally, we want to thank the publishing house, World Scientific, for making possible the interesting contents of our seminars to be enjoyed by all the mathematical community.

The Editors

Contents

This page intentionally left blank

INTRODUCTION TO THE UNIFORM CLASSIFICATION OF BANACH SPACES

YOAV BENYAMINI*
Department of Mathematics, Technion – Israel Institute of Technology
Haifa 32000, Israel
e-mail: yoavb@tx.technion.ac.il

This is an introductory survey of the classification of Banach spaces as metric spaces, where the maps are (nonlinear) uniformly continuous maps or, more specifically, Lipschitz maps. We describe basic results which show that the uniform theory and the linear theory are different but that, nevertheless, some linear features of a Banach space are preserved under uniform homeomorphisms.

1. Introduction

The norm on a Banach space induces on it a topology and a metric. In the linear theory of Banach spaces the natural maps are continuous linear maps, and these maps "respect" not only the topology but also the metric structure: they are bounded, i.e., Lipschitz maps. Of special interest are the linear maps which preserve the metric, i.e., isometries.

One can, however, disregard the linear structure of the Banach spaces and consider them as a special class of topological spaces or as a special class of metric spaces. In this setup the maps are no longer required to be linear and we consider continuous maps in the first case and uniformly continuous, Lipschitz or isometric maps in the second.

The following two theorems define the "natural boundaries" for a meaningful theory. The first is due to Mazur and Ulam [32] and the second to Kadec [26] (for separable Banach spaces) and to Toruńczyk [39] (for general density characters).

Theorem 1.1. *Let E and F be Banach spaces. If $f : E \to F$ is a surjective isometry satisfying $f(0) = 0$, then f is linear.*

*This work was supported by the Technion Fund for the Promotion of Research.

Theorem 1.2. *Two infinite-dimensional Banach spaces are homeomorphic to each other iff they have the same density character.*

These two theorems say that the topological and isometric classifications of Banach spaces are "trivial" in two opposite ways: Theorem 1.1 implies that the linear structure can be completely recovered from the metric structure and thus reduces the (seemingly) nonlinear isometric classification problem to the linear one. Theorem 1.2 says that the topology gives no information whatsoever on the linear structure.

The uniform classification of Banach spaces is "inbetween" these two extreme theories. As we shall see in these notes this is an interesting theory which, on the one hand, gives some information on the linear structure of the spaces but, on the other hand, it is rich enough to be genuinely different from the linear theory.

A word of warning before we go on. While the topological or nonlinear isometric theories are "trivial" from the point of view of classification, they are certainly far from being trivial. Infinite-dimensional topology is a rich theory, see the books by Bessaga and Pełczyński [10] and van Mill [40]. Similarly, there are, of course, many problems concerning isometric (or, more generally, 1-Lipschitz) maps, such as into-isometries, extensions, nonexpansive retractions, fixed points etc., which we do not consider here at all.

The purpose of this survey is to describe the main results, ideas and techniques in the area. The idea is to draw the attention of the reader to this exciting area and to stimulate further reading and interest. We did not try to give "best" results or to analyze variations of the concepts and techniques. For these and for many more topics that are not discussed here at all we refer the reader to the recent book [8] by the author and J. Lindenstrauss. We also refer to [8] for unexplained notation and terminology.

The article is divided to three sections. The first is a general description of the area in the spirit of an expanded colloquium talk. In the other two sections we describe in more detail and with proofs, or sketches of proofs, some of the main results on the Lipschitz classification (in Section 3) and on the uniform classification (in Section 4) of Banach spaces.

These notes are an expanded version of lectures delivered at the "First International Course of Mathematical Analysis in Andalucia", which was held in Cadiz, Spain, in September 2002. It is a pleasure to thank the organizers and the participants of the course for the pleasant and inspiring atmosphere. I also thank the referee for his useful comments.

2. A General Overview

Historically, the first result in infinite-dimensional topology is the following theorem of Mazur [31]. Denote the closed unit ball of $L_p(\mu)$ by $B_p(\mu)$.

Theorem 2.1. *Let μ and ν be two measures and let $1 \le p, r < \infty$. If $L_p(\mu)$ and $L_r(\nu)$ have the same density character, then their unit balls, $B_p(\mu)$ and $B_r(\nu)$, are uniformly homeomorphic to each other.*

The homeomorphism of $B_p(\mu)$ onto $B_r(\mu)$ (for the same measure μ) is given explicitly by the map $f \to |f|^{p/r}\text{sign}(f)$. For different measures we first use these maps pass to $B_2(\mu)$ and $B_2(\nu)$ and then note that these balls are even isometric to each other.

Note that the L_p spaces are not linearly isomorphic to each other for different values of p. Hence the uniform classification (at least of the unit balls of Banach spaces) is different from the linear classification of the spaces. We shall see later that L_p and L_q (for $p \ne q$), as well as L_p and l_p (for $p \ne 2$) are also not uniformly homeomorphic to each other. Thus the uniform classification of subsets of Banach spaces is also different from the uniform classification of the whole spaces.

The next result, which generalizes Mazur's theorem, was proved for spaces with an unconditional basis by Odell and Schlumprecht [35] and for general lattices by Chaatit [12]. For proofs and more details see also Sections 9.1-9.3 in [8].

Theorem 2.2. *Let E be a Banach lattice which does not contain l_∞'s uniformly. Then its unit ball is uniformly homeomorphic to the unit ball of a Hilbert space.*

The condition that E does not contain l_∞'s uniformly is, in fact, also necessary. This was proved much earlier by Enflo [15].

Aharoni and Lindenstrauss [2] gave the first example of two Lipschitz equivalent Banach spaces which are not isomorphic to each other. It follows that the Lipschitz classification of Banach spaces is different from their linear classification.

We now describe the structure of the example. Let $q : E \to F$ be a linear quotient map from E onto F with kernel Z. A linear operator $T : F \to E$ is called a lifting of q if $qT = id_F$. Such a lifting is an isomorphism of F into E whose image, TF, is complemented in E (by the projection Tq). The kernel Z is complemented in E (by the projection $id_E - Tq$) and E is isomorphic to $F \oplus Z$. If T is a (nonlinear) Lipschitz lifting, then the same formulas yield similar Lipschitz consequences: F is Lipschitz equivalent to a Lipschitz retract of E, the kernel Z is a Lipschitz retract of E and E is Lipschitz equivalent to $F \oplus Z$.

The example of Aharoni and Lindenstrauss is a Banach space E which does not contain a subspace isomorphic to $c_0(\Gamma)$ (for some uncountable set Γ), but such that there is a surjective linear quotient map $q : E \to c_0(\Gamma)$ with kernel c_0 and such that q admits a Lipschitz lifting. It follows that E is Lipschitz equivalent to $c_0(\Gamma) \oplus c_0 = c_0(\Gamma)$, but they are certainly not isomorphic to each other: E does not even contain an isomorphic copy of $c_0(\Gamma)$.

Godefroy and Kalton [17] have recently introduced a "categorical" construction, which, among other things, gave a systematic method of constructing more examples of the same nature. (See also Kalton [27] where the approach is extended to spaces of Hölder and more general uniformly continuous functions.) We now describe the construction in [17].

Let E be a Banach space and denote by $Lip_0(E)$ the Banach space of Lipschitz functions $f : E \to \mathbb{R}$ satisfying $f(0) = 0$. The norm of f is its Lipschitz constant. Let $\mathcal{F}(E)$ denote the closed linear subspace of the dual $Lip_0(E)^*$, spanned by the point evaluations $\delta_x(f) = f(x)$. The space $\mathcal{F}(E)$ is called the Lipschitz free space over E. Lipschitz maps between Banach spaces induce, in a natural way, linear maps between their free spaces and $\mathcal{F}(E)$ enjoys some useful properties of free objects in a category. The map $\delta_E : E \to \mathcal{F}(E)$ is a (nonlinear!) isometry of E into $\mathcal{F}(E)$. One checks easily that the linear map $\beta_E : \mathcal{F}(E) \to E$, given by $\beta_E(\sum a_n \delta_{x_n}) = \sum a_n x_n$, is a surjective quotient map with $\|\beta_E\| = 1$. Clearly δ_E is an isometric lifting of β_E, hence $\mathcal{F}(E)$ is Lipschitz equivalent to $E \oplus \ker(\beta_E)$.

The next theorem yields the required examples. Recall that a Banach space E is weakly compactly generated (WCG) if there is a weakly compact subset $K \subset E$ whose linear span is dense in E. Since the unit ball of a reflexive space is weakly compact, reflexive spaces are certainly WCG.

Theorem 2.3. *Let E be a nonseparable WCG space. Then $\mathcal{F}(E)$ does not contain a subspace isomorphic to E. In particular $\mathcal{F}(E)$, which is Lipschitz equivalent to $E \oplus \ker(\beta_E)$, is not isomorphic to it.*

All the examples known to date of Lipschitz equivalent Banach spaces which are not isomorphic to each other are nonseparable and nonreflexive. The main open problem in this area is

Problem 2.1. *If two separable Banach spaces are Lipschitz equivalent to each other, are they necessarily isomorphic? What if they are also reflexive? Uniformly convex?*

It turns out that the method used to construct nonseparable examples cannot work in the separable case. This is one consequence of the following remarkable theorem of Godefroy and Kalton [17].

Theorem 2.4. *Let F be a separable Banach space and assume that it is the image of some Banach space E under a surjective quotient map $q : E \to F$. If q admits a Lipschitz lifting, then it also admits a linear lifting.*

In the linear theory the space c_0 is considered to be a "small" space. For example, it does not contain reflexive subspaces or any of the other classical Banach spaces. This is no longer true in the Lipschitz category. Aharoni [1] proved that it is actually "universal". The proof of the following theorem is presented in Section 3.

Theorem 2.5. *Every separable metric space is Lipschitz equivalent to a subset of c_0.*

While we do not know whether the Lipschitz and linear classifications of separable Banach spaces coincide, the uniform classification is certainly different from both. The following result is due to Aharoni and Lindenstrauss [3], improving on a previous result of Ribe [38]. For a proof, more details and extensions see Section 10.4 in [8].

Theorem 2.6. *Let $1 \leq p, q, p_n < \infty$ with $p_n \to p$ and let $E = (\sum \oplus l_{p_n})_q$. Then E is uniformly homeomorphic to $E \oplus l_p$. If $p \neq q$ and $p_n \neq p$ for all n, then E and $E \oplus l_p$ are not isomorphic. If $p = 1$ and $q, p_n > 1$ for all n, then E is reflexive while $E \oplus l_1$ is not, hence (see Corollary 2.1(i) below) they are not even Lipschitz equivalent.*

The results up to this point showed the difference between the uniform and the linear classification. We now discuss some results in the opposite direction, namely, instances in which at least some of the linear structure is preserved.

The most natural way to "linearize" a mapping is by differentiation. Recall that a mapping $f : E \to F$ is said to be Gâteaux differentiable at a point x if the limit $Du = \lim_{t \to 0} \big(f(x + tu) - f(x)\big)/t$ exists for every $u \in E$ and is a bounded linear operator as a function of u. The operator D is called the Gâteaux derivative of f at x and is denoted by $D_f(x)$.

If f is a Lipschitz mapping with constant K, which is Gâteaux differentiable at some point x, then the derivative $D = D_f(x)$ is bounded by the same constant K. Moreover, if f is a Lipschitz equivalence (i.e., a Lipschitz map which also satisfies the lower estimate $\|f(x) - f(y)\| \geq \|x - y\|/K$ for every $x, y \in E$), then D is also bounded from below by the same constant, and is thus an into-isomorphism. (Note that $D_f(x)$ is only bounded from below. Its image may very well be a proper subspace of F even when f was assumed to be a surjective Lipschitz equivalence.) It follows from this discussion and from the next theorem that in many cases the existence of a Lipschitz embedding of E into F implies that there is also a linear isomorphism of E into F. (The theorem, due independently to Aronszajn [5], Christensen [13] and Mankiewicz [30], will be discussed in detail in Section 3.)

Theorem 2.7. *Let E be a separable Banach space and assume F has the Radon-Nikodým property (RNP). If $f : E \to F$ is a Lipschitz function, then there is a point $x \in E$ where f is Gâteaux differentiable.*

It is obvious that some assumption is needed in Theorem 2.7 on the space F. For example, one cannot take $F = c_0$ because by Theorem 2.5 every separable Banach space is Lipschitz embedable in c_0, but "most" Banach spaces are not linearly embedable in it. The assumption that F has RNP is actually essential. Indeed, one of the characterizations of RNP is that F has RNP iff every Lipschitz map from $\mathbb{R}$ to F is differentiable at some point of $\mathbb{R}$ (or, equivalently, differentiable almost everywhere). Thus for the theorem to hold we must assume that F has RNP.

Since reflexive spaces have RNP we obtain

Corollary 2.1. *(i) If a separable Banach space E is Lipschitz equivalent to a subset of a reflexive Banach space F, then E is isomorphic to a subspace of F.*

(ii) If E is Lipschitz equivalent to a subset of a Hilbert space, then it is isomorphic to a Hilbert space.

(iii) If $p > 1$ and $r \geq 1$, then L_r is Lipschitz equivalent to a subset of L_p iff it is isomorphic to a subspace of L_p, i.e., iff $r = 2$ or $2 \geq r \geq p$.

Part (ii) was originally proved by Enflo [16]. To deduce it from Theorem 2.7 we need the fact that E is isomorphic to a Hilbert space iff all its separable subspaces are. Part (iii) also holds for $p = 1$, but this requires an additional argument.

Differentiation results are, of course, not available for general uniformly continuous maps. We now describe some of the techniques used in their study. More details, including the definition and some basic facts on ultraproducts, will be given in Section 4.

A basic useful property of uniformly continuous mappings is that they satisfy a Lipschitz condition for large distances. More precisely

Proposition 2.1. *Let $f : E \to F$ be uniformly continuous. Then for every $\alpha > 0$ there is a constant $K = K(\alpha) > 0$ such that $\|f(x)-f(y)\| \leq K\|x-y\|$ whenever $\|x - y\| \geq \alpha$.*

Proof. Fix α and let $A = \sup\{\|f(x)-f(y)\| : \|x-y\| \leq \alpha\}$. If $\|x-y\| \geq \alpha$, choose $N \sim \|x - y\|/\alpha$ and divide the interval $[x, y]$ to N subintervals $[x_{j-1}, x_j]$ with $\|x_j - x_{j-1}\| \leq \alpha$. Then

$$\|f(x) - f(y)\| \leq \sum_{j \leq N} \|f(x_j) - f(x_{j-1})\| \leq AN \sim A\|x - y\|/\alpha$$

so take $K(\alpha) \sim A/\alpha$. □

One way to apply the proposition is to use it to create Lipschitz maps from uniformly continuous ones: Assume that $f : E \to F$ is uniformly continuous and assume, as in the proposition, that $\|f(x)-f(y)\| \leq K\|x-y\|$ whenever $\|x - y\| \geq 1$. Put $f_n(x) = f(nx)/n$. Then the map f_n satisfies $\|f_n(x) - f_n(y)\| \leq K\|x - y\|$ whenever $\|x - y\| \geq 1/n$. It follows that if $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$, then $g = (f_n)_{\mathcal{U}}$ is a K-Lipschitz map from the ultrapower $(E)_{\mathcal{U}}$ into $(F)_{\mathcal{U}}$. Moreover, if f is a uniform homeomorphism, then one can apply the same procedure to f^{-1}. One checks directly that $(f_n)^{-1} = (f^{-1})_n$, thus the procedure gives a Lipschitz inverse to g. We have thus proved the following theorem of Heinrich and Mankiewicz [21]

Theorem 2.8. *If E and F are uniformly homeomorphic, then they have Lipschitz equivalent ultrapowers.*

As we shall see in Section 4, this result, together with the results of Section 3, combine to give a simple proof of the following theorem of Ribe [37]. Roughly speaking, the theorem says that uniformly homeomorphic Banach spaces have "the same" finite-dimensional subspaces or, in the language of Banach spaces theory, that they have the same local linear structure.

Theorem 2.9. *Let E and F be two uniformly homeomorphic Banach spaces. Then they are crudely finitely-representable in each other, i.e., there* *is a constant $C > 0$ so that for every finite-dimensional subspace E_1 of E there is a finite-dimensional subspace $F_1 \subset F$ with $d(E_1, F_1) \le C$ and vice versa.*

In particular, since L_p and L_r do not have the same local structure when $p \neq r$ (as follows, for example, by computing their type and cotype), it follows that they are not uniformly homeomorphic to each other.

Proposition 2.1 can also be applied through the study of approximate midpoints.

Definition 2.1. Fix $x, y \in E$ and $\delta > 0$. Then the set of δ-approximate midpoints between x and y is

$$\mathrm{Mid}(x, y, \delta) = \{z : \; \|x - z\|, \|y - z\| \le (1 + \delta)\|x - y\|/2\}$$

When $\delta = 0$ we say that z is a midpoint (or, when we want to emphasize, exact midpoint) between x and y.

It is clear that exact midpoints are mapped by isometries to exact midpoints. More generally, if f is K-Lipschitz and two points x and y happen to satisfy $\|f(x) - f(y)\| = K\|x - y\|$, then f maps exact midpoints between x and y to exact midpoints between their images. The following proposition generalizes this fact.

Proposition 2.2. *Let $f : E \to F$ be a uniform homeomorphism and let $0 < \delta < 1/2$. Then there are points x, y with $\|x - y\|$ arbitrarily large, so that*

$$f\big(\mathrm{Mid}(x, y, \delta)\big) \subset \mathrm{Mid}\big(f(x), f(y), 5\delta\big).$$

Proof. We only sketch the proof and leave the exact computations, that give the estimate 5δ, to the reader.

Let $K(\alpha)$ be the Lipschitz constant of f for distances larger than α. Then $K(\alpha)$ is a nonincreasing function of α. Put $K = \lim_{\alpha \to \infty} K(\alpha)$. Since f^{-1} is also uniformly continuous, it follows that $K > 0$. Fix α so large that $K(\alpha/2) \sim K$. Choose x, y with $\|x - y\| > \alpha$ such that $\|f(x) - f(y)\| \sim K\|x - y\|$ and let $z \in \mathrm{Mid}(x, y, \delta)$. Then

$$\begin{aligned}\|f(x) - f(z)\| &\le K(\alpha/2)\|x - z\| \sim K\|x - z\| \\ &\le (1 + \delta)K\|x - y\|/2 \sim (1 + \delta)\|f(x) - f(y)\|/2.\end{aligned}$$

Similarly $\|f(y) - f(z)\| \preceq (1 + \delta)\|f(x) - f(y)\|/2$. □

We now explain the idea of how the approximate midpoint sets can be used to show that some Banach spaces E and F cannot be uniformly homeomorphic to each other: Compute the approximate midpoint sets for points in E and in F. By the proposition a uniform homeomorphism $f : E \to F$ will have to take some approximate midpoint set in E into an approximate fixed point set in F. If it so happens that approximate midpoints sets in E are "large" sets and approximate midpoint sets in F are "small", then this would contradict the uniform continuity of f^{-1}. To demonstrate how this idea is implemented we shall compute in Section 4 the approximate midpoint sets in l_p and L_p, and then use the right notions of "large" and "small", appropriate for the different situations, to prove

Theorem 2.10. *For every $1 \leq p < \infty$, $p \neq 2$, the spaces L_p and l_p are not uniformly homeomorphic to each other.*

This method was introduced by Enflo (unpublished), who used it to prove the case $p = 1$ of the theorem. The case $1 < p < 2$ is due to Bourgain [11] and the case $p > 2$, which required a new notion of "large" and "small" is due to Gorelik [19].

The problem of characterizing Banach spaces which are uniformly homeomorphic to a subset of a Hilbert space was completely solved by Aharoni, Maurey and Mityagin [4]. The result is, in fact, more general and holds for linear metric spaces and not only for Banach spaces. For example, it follows from the next theorem and from known results in the linear theory that $L_p(\mu)$ is uniformly homeomorphic to a subset of a Hilbert space iff $0 \leq p \leq 2$.

Theorem 2.11. *A real linear metric space is uniformly homeomorphic to a subset of a Hilbert space iff it is linearly isomorphic to a subspace of $L_0(\mu)$ for some measure μ.*

We shall not discuss this theorem in these notes. The interested reader is referred to Chapter 8 in [8].

The last problem that we discuss in this section is the uniform and Lipschitz classification of balls and spheres in Banach spaces.

Benyamini and Sternfeld [9] proved that for every infinite-dimensional Banach space E there is a Lipschitz retraction from the unit ball $B(E)$ onto the unit sphere $S(E)$. Equivalently, $S(E)$ is Lipschitz contractible and there is a Lipschitz map on $B(E)$ with no approximate fixed point. (The results in [9] followed Nowak [34], who proved them for some special spaces. See also Azagra and Cepedello-Boiso [7] for a smooth version of these results and for generalizations to starlike sets.)

This result leads naturally to the following open problem.

Problem 2.2. *Let E be an infinite-dimensional Banach space, are its unit ball $B(E)$ and unit sphere $S(E)$ Lipschitz equivalent? Are they uniformly equivalent?*

The problem is open for any Banach space (including Hilbert space!) except for one "pathological" counter-example: Gowers and Maurey [20] constructed a separable reflexive Banach space, GM, which is not isomorphic to any of its subspaces. For this space the unit ball cannot be Lipschitz equivalent to the unit sphere. Indeed, if $f : B \to S$ were a Lipschitz equivalence, then since the space is separable and reflexive f would be Gâteaux differentiable at some point by Theorem 2.7. The derivative, D, would be an isomorphism from GM into a proper subspace of itself (as is easy to deduce from the fact that the image of f is in the sphere, which "looks" locally like a subspace of codimension one). But there is no isomorphism of GM onto a proper subspace of itself!

It is easy to see that for any Banach space E the unit sphere $S(E)$ is Lipschitz homogeneous, i.e., for any two points x, y in the sphere there is a Lipschitz homeomorphism of the sphere onto itself taking x to y. It follows that if the ball and the sphere of E are Lipschitz (or uniformly) equivalent, then the ball should also be Lipschitz (or uniformly) homogeneous. The only nontrivial advance on Problem 2.2 is the following result of Nahum [33]. See Section 9.4 in [8] for a proof and further discussion.

Theorem 2.12. *Let E be a Banach space which is isomorphic to $E \oplus \mathbb{R}$, then $B(E)$ is Lipschitz (respectively, uniformly) equivalent to $S(E)$ iff it is Lipschitz (respectively, uniformly) homogeneous.*

Here is a simple application of the theorem.

Corollary 2.2. *Let E and F be two Banach spaces which are isomorphic to $E \oplus \mathbb{R}$ and $F \oplus \mathbb{R}$ respectively. Then the balls $B(E)$ and $B(F)$ are Lipschitz (respectively, uniformly) equivalent to each other iff the spheres $S(E)$ and $S(F)$ are.*

Proof. It is clear that if f is an equivalence between the spheres, then its homogeneous extension gives an equivalence between the balls. Conversely, assume that $f : B(E) \to B(F)$ is a Lipschitz (or uniform) equivalence. If f

takes $S(E)$ onto $S(F)$, then we are done. So assume that there is a point $x \in S(E)$ such that $\|f(x)\| < 1$, and choose a point $y \in E$ with $\|y\| < 1$ and $\|f(y)\| < 1$. (Any point y with $\|y\| < 1$ which is close enough to x certainly satisfies this condition.) Let g be a Lipschitz homeomorphism of $B(F)$ onto itself which takes $f(x)$ onto $f(y)$. Then $f^{-1} \circ g \circ f$ maps the point x in $S(E)$ to the point $y \in E$ with $\|y\| < 1$. Since pairs of points in $S(E)$ can be mapped to each other by a Lipschitz equivalence of $B(E)$ and the same is true for pairs in $B(E) \setminus S(E)$, it follows that $B(E)$ is homogeneous and the same is true for $B(F)$ which is Lipschitz (or uniformly) homeomorphic to it. By the theorem $B(E)$ and $B(F)$ are Lipschitz (or uniformly) homeomorphic to $S(E)$ and $S(F)$ repectively. Thus all the four sets involved are equivalent to each other and, in particular, so are $S(E)$ and $S(F)$. □

3. Lipschitz Maps

The main result in this section will be the proof of Theorem 2.7 as well as some further comments and variations. But we start with a proof, essentially due to Assouad [6], of Theorem 2.5.

Proof of Theorem 2.5. A map $f : X \to c_0$ is given by a sequence of real-valued functions $f(x) = (f_1(x), f_2(x), ...)$, where the f_n's satisfy

(i) $f_n(x) \to 0$ for every $x \in X$.

When X is a metric space, then f is a Lipschitz map iff

(ii) The f_n's are Lipschitz with a common Lipschitz constant.

And f is a Lipschitz embedding if also

(iii) There is a constant $C > 0$ so that for every $x \neq y$ in X there is a n such that $|f_n(x) - f_n(y)| \geq C\|x - y\|$.

In the construction we shall use functions of the form $\big(a - d(x, M)\big)^+$ for suitably chosen sets $M \subset X$ and constants a. Such functions are Lipschitz with constant 1.

To present the idea of the proof in a somewhat simpler setup, let us first assume that X is compact and that its diameter is 1.

For each $n \geq 0$ let $\{x_i^n : i \leq m_n\}$ be a finite 2^{-n}-net in X and put $f_{n,i}(x) = \big(2^{-n+1} - d(x, x_i^n)\big)^+$. Then (i) and (ii) hold because $|f_{n,i}(x)| \leq 2^{-n+1} \to 0$ and $\|f_{n,i}\|_{Lip} = 1$. To check (iii) fix $x \neq y$. Let n satisfy $2^{-(n+1)} \leq d(x, y) \leq 2^{-n}$ and choose i such that $d(x, x_i^{n+3}) < 2^{-(n+3)}$. Then

$$\begin{aligned} f_{n+3,i}(x) &= 2^{-(n+3)+1} - d(x, x_i^{n+3}) \\ &\geq 2^{-(n+3)+1} - 2^{-(n+3)} = 2^{-(n+3)} \geq d(x, y)/8. \end{aligned}$$