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Algebraic Cycles, Sheaves, Shtukas, and Moduli

Impanga Lecture Notes

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Preface

The articles in this volume are an outgrowth of seminars and schools of Impanga in the period 2005–2007. Impanga is an algebraic geometry group operating since 2000 at the Institute of Mathematics of Polish Academy of Sciences in Warsaw. The present volume covers, besides seminars, the following schools organized by Impanga at the Banach Center in Warsaw:

- *Moduli spaces*, April 2005,
- *Algebraic cycles and motives*, October 2005,
- *A tribute to Hoene-Wroński*, January 2007.

More information about Impanga, including complete lists of seminars, schools, and sessions, can be found at the web-page:

$$\texttt{http}:// \texttt{www.impan.gov.pl}/\sim \texttt{pragacz/impanga.htm} \ .$$

Let us describe briefly the contents of the lecture notes in this volume. [1]

Jean-Marc Drézet, in his first article, discusses fine moduli spaces of coherent sheaves, i.e., those endowed, at least locally, with universal sheaves. Whereas the most known fine moduli spaces appear in the theory of (semi)stable sheaves, the author constructs other, the so called "exotic" fine moduli spaces; the corresponding sheaves are sometimes not simple.

The subject of the second article of Jean-Marc Drézet is the study of moduli spaces of coherent sheaves on multiple curves embedded in a smooth projective surface. The author introduces new invariants for such curves: canonical filtrations, generalized rank and degree, and proves a Riemann-Roch theorem. A more detailed study of coherent sheaves on double curves is presented.

Tomas L. Gomez gives an outline of constructions of different moduli spaces. His starting point is the Jacobian of a smooth projective curve, and the final aims are moduli spaces of principal sheaves. A pretty complete account of the theory of principal bundles and sheaves is presented; a special emphasis is put on their stability properties. Orthogonal and symplectic sheaves serve as instructing examples.

[1] The lecture notes by J.-M. Drézet, T.L. Gomez, A.H.W. Schmitt, and Ngo Dac Tuan stem from the first school, the article by V. Srinivas from the second school, the opening article of P. Pragacz from the third school, and finally the articles by A. Langer, P. Pragacz, and that by P. Pragacz-A. Weber from the seminars of Impanga.

Adrian Langer gives a comprehensive introduction to torsion free sheaves and the moduli spaces of (semi)stable sheaves in any dimension and arbitrary characteristic. The author discusses carefully the (semi)stability conditions and restriction theorems. One of the main goals is to give the boundedness results, which are crucial to construct moduli spaces using the techniques of the Quot-schemes. Line bundles on the moduli spaces are also described, and generic smoothness of the moduli spaces of sheaves on surfaces is showed.

Piotr Pragacz discusses some topological, algebraic, and geometric properties of the zero schemes of sections of vector bundles, namely the connectedness and the “point” and “diagonal” properties. An overview of recent results by Vasudevan Srinivas, Vishwambhar Pati, and the author on these properties is presented.

Piotr Pragacz and Andrzej Weber generalize Thom polynomials from singularities of maps to invariant cones in representations of products of linear groups. With the help of the Fulton-Lazarsfeld theory of positivity of ample vector bundles, they show that the coefficients of Thom polynomials expanded in the basis of the products of the Schur functions, are nonnegative.

Alexander H.W. Schmitt gives an account of classical and new results in Geometric Invariant Theory (especially the theory relative to a base curve), and present a recent progress in the construction of moduli spaces of vector bundles and principal bundles with extra structure (called augmented or “decorated” vector or principal bundles). The problems of taking various quotients and stability conditions are widely discussed and illustrated by numerous examples.

Vasudevan Srinivas shows some applications of the intersection theory of algebraic cycles to commutative algebra. A special emphasis is put on the study of the groups of zero-dimensional cycles, modulo rational equivalence, on smooth projective or affine varieties (in particular, surfaces). Their applications to embedding and immersion of affine varieties, indecomposable projective modules, and the complete intersection property are given.

Ngo Dac Tuan presents a “friendly” introduction to shtukas, the stacks of shtukas, and their compactifications. The notion of a “shtuka” was first introduced by Drinfeld and used in his proof of the Langlands correspondence for GL_2 over function fields. It recently has been used by Lafforgue in his proof of the Langlands correspondence for higher groups GL_r over function fields.

We dedicate the whole volume to the memory of **Józef Maria Hoene-Wroński** – one of the most original figures in the history of science. The opening article by Piotr Pragacz discusses some aspects of his life and work.

Acknowledgments. The Editor thanks the authors for their scientific contributions, to Adrian Langer and Halszka Gasińska-Tutaj for their help with the school on moduli spaces, and finally to Dr. Thomas Hempfling from Birkhäuser-Verlag for a pleasant editorial cooperation.

Warszawa, July 2007 *The Editor*

Algebraic Cycles, Sheaves, Shtukas, and Moduli
Trends in Mathematics, 1–20

Notes on the Life and Work of Józef Maria Hoene-Wroński

Piotr Pragacz

To reach the source, one has to swim against the current.
Stanisław J. Lec

Abstract. This article is about Hoene-Wroński (1776–1853), one of the most original figures in the history of science. It was written on the basis of two talks delivered by the author during the session of Impanga "A tribute to Józef Hoene-Wroński"[1], which took place on January 12 and 13, 2007 in the Institute of Mathematics of the Polish Academy of Sciences in Warsaw.

1. Introduction and a short biography. This article is about Józef Maria Hoene-Wroński. He was – primarily – an uncompromising searcher of truth in science. He was also a very original philosopher. Finally, he was an extremely hard worker.

When reading various texts about his life and work and trying to understand this human being, I couldn't help recalling the following motto:

> *Learn from great people great things which they have taught us. Their weaknesses are of secondary importance.*

A short biography of Józef Maria Hoene-Wroński:

1776	–	born on August 23 in Wolsztyn;
1794	–	joins the Polish army;
1795–1797	–	serves in the Russian army;
1797–1800	–	studies in Germany;
1800	–	comes to France and joins the Polish Legions in Marseilles;
1803	–	publishes his first work *Critical philosophy of Kant*;
1810	–	marries V.H. Sarrazin de Montferrier;
1853	–	dies on August 9 in Neuilly near Paris.

Translated by Jan Spaliński. This paper was originally published in the Polish journal *Wiadomości Matematyczne* (*Ann. Soc. Math. Pol.*) vol. 43 (2007). We thank the Editors of this journal for permission to reprint the paper.

One could say that the starting point of the present article is chapter XII in [6]. I read this article a long time ago, and even though I read a number of other publications about Hoene-Wroński, the content of this chapter remained present in my mind due to its balanced judgments. Here we will be mostly interested in the mathematics of Wroński, and especially in his contributions to algebra and analysis. Therefore, we shall only give the main facts from his life – the reader may find more details in [9]. Regarding philosophy, we shall restrict our attention to the most important contributions – more information can be found in [36], [37], [47], and [10]. Finally, Wroński's most important technical inventions are only mentioned here, without giving any details.

Józef Maria Hoene-Wroński
(daguerreotype from the Kórnik Library)

2. Early years in Poland. Józef Hoene was born in Wolsztyn on August 23[1] 1776. His father, Antoni, was a Czech imigrant and a well-known architect. A year later the family moved to Poznań, where the father of the future philosopher became a famous builder (in 1779 Stanisław August – the last King of Poland – gave him the title of the *royal architect*). In the years 1786–1790 Józef attended school in Poznań. Influenced by the political events of the time, he decided to join the army. His father's opposition was great, but the boy's determination was even greater. (Determination is certainly the key characteristic of Wroński's nature.) In 1792 he run away from home and changed his name, to make his father's search more difficult. From that time on he was called Józef Wroński and under this name he was drafted by the artillery corps. In the uprising of 1794 he was noted for

[1]Various sources give the 20 and the 24 of August.

his bravery, and was quickly promoted. During the defense of Warsaw against the Prussian army he commanded a battery – and was awarded a medal by commander in chief Tadeusz Kościuszko for his actions. He also took part in the battle near Maciejowice, during which he was taken to captivity. At that time he made the decision to join the Russian army. What was the reason for such a decision we do not know; while consulting various materials on the life of Wroński, I haven't found a trace of an explanation. Maybe – this is just a guess – he counted on the possibility of gaining an education in Russia: Wroński's main desire was a deep understanding of the laws of science, and these are universal: the same in Russia as elsewhere.... After being promoted to the rank of captain, he became an advisor of the General Staff of Suworow. In the years 1795–1797 he serves in the Russian army and is promoted to the rank of lieutenant-colonel.

3. Departure from Poland. The information about the sudden death of his father changed Wroński's plans. He inherited a large sum, which allowed him to devote himself to his studies, as he wanted for a long time. He quit the army and travelled West. Greatly inspired by Kant's philosophy, he arrived at Königsberg. However, when he found out that Kant is no longer giving lectures, Wroński left for Halle and Göttingen. In 1800 he visited England, and afterwards came to France. Fascinated by Dąbrowski's Legions, he asked the general for permission to join them. Dąbrowski agreed (however he did not honor the rank Wroński gained in Tzar's army) and sent him to Marseilles. There, Wroński could combine his service with his love for science. He became a member of Marseilles' Academy of Science and Marseilles' Medical Society.

In Marseilles Wroński underwent an enlightenment. This turning point in his life was a vision, which he had on August 15, 1803 at a ball on Napoleon's birthday. As he had described it, he had a feeling of anxiety and of certainty, that he would discover the "essence of the Absolute". Later he held that he understood the mystery of the beginning of the universe and the laws which govern it. From that time on he decided to reform human thought and create a universal philosophical system. In remembrance of that day he took the name of Maria and went down in history of science as Józef Maria Hoene-Wroński. Wroński's reform of human knowledge was to be based on a deep reform of mathematics by discovering its fundamental laws and methods. At the same time he posed the problem of solving the following three key issues in (applied) mathematics:

1. Discovering the relation between matter and energy (note Wroński's incredibly deep insight here);
2. the formation of celestial objects;
3. the formation of the universe from the celestial objects.

The most visible characteristic of Wroński's work is his determination to base all knowledge on philosophy, by finding the general principle, from which all other knowledge would follow.

Resources needed to publish papers have quickly run out, and Wroński started to support himself by giving private lessons of mathematics. Among his students was Victoria Henriette Sarrazin de Montferrier. The teacher liked this student so much, that in 1810 she became his wife. In September of the same year Wroński heads for the conquest of Paris.

4. Paris: solving equations, algorithms, continued fractions and struggles with the Academy. In 1811 Wroński publishes *Philosophy of Mathematics* [14] (see also [24]). Even earlier he has singled out two aspects of mathematical endeavor:

1. theories, whose aim is the study of the essence of mathematical notions;
2. algorithmic techniques, which comprise all methods leading to the computation of mathematical unknowns.

The second point above shows that Wroński was a pioneer of "algorithmic" thinking in mathematics. He gave many clever algorithms for solving important mathematical problems.

In 1812 Wroński publishes an article about solving equations of all degrees [15] (see also [20]). It seems that, without this paper, Wroński's scientific position would be clearer. In this paper Wroński holds that he has found *algebraic* methods to find solutions of equations of arbitrary degree. However, since 1799 it has been believed that Ruffini has proved the impossibility of solving equations of degree greater than 4 by radicals (Ruffini's proof – considered as essentially correct nowadays – at that time has lead to controversy[2] and the mathematical community has accepted this result only after Abel has published it in 1824). So did Wroński question the Ruffini–Abel theorem? Or did he not know it? As much as in the later years Wroński really did not systematically study the mathematical literature, in the first decade of the nineteenth century he has kept track of the major contributions. If one studies carefully the (difficult to understand) deliberations and calculations, it seems that, Wroński's method leads to approximate solutions, in which the error can be made arbitrarily small[3]. In his arguments besides algebraic methods, we find analytic and transcendental ones. This is the nature, for example, of his solution of the factorization problem coming from the work cited above, which we describe below. This type of approach is not quite original, it has been used by Newton for example[4]. Since Wroński considered this work so important (it was reprinted again towards the end of 1840) – in order to gain a true picture of the situation – it would be better to publish a new version with appropriate comments of someone competent, explaining what Wroński does and what he *does not* do.

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[2]Ruffini published his results in a book and in 1801 sent a copy to Lagrange, however he did not receive any response. Legendre and other members of the Paris Academy did not consider this work as worthy of attention. Only in 1821 – a year before his death – Ruffini received a letter from Cauchy who wrote that he considers Ruffini's result as very important.

[3]It is interesting that the authors of [6] have reached a similar opinion, but without further details.

[4]Methods of Newton–Raphson and Laguerre are known.

I think that such a competent person could have been Alain Lascoux, who, reading this and other works of Wroński, could see, that he was addressing the following three algebraic problems, connected with polynomials of one variable and Euclid's algorithm for such polynomials:

1. Consider two normed polynomials $F(x)$ and $G(x)$. Suppose that $\deg(F) \geq \deg(G)$. Performing multiple division of $F(x)$ and $G(x)$:

$$F = * \, G + c_1 R_1, \quad G = * \, R_1 + c_2 R_2, \quad R_1 = * \, R_2 + c_3 R_3, \quad \ldots .$$

Successive coefficients " $*$ " are uniquely determined polynomials of the variable x such that

$$\deg G(x) > \deg R_1(x) > \deg R_2(x) > \deg R_3(x) > \cdots .$$

Instead of the "ordinary" Euclid's algorithm, were $c_1 = c_2 = c_3 = \cdots = 1$ and where $R_i(x)$ are *rational* functions of the variable x and roots of $F(x)$ and $G(x)$, one can choose c_i in such a way that the successive remainders $R_i(x)$ are polynomials of the variable x and of those roots. These remainders are called *normed polynomial remainders* or *subresultants*. Wroński constructed a clever algorithm for finding $R_i(x)$ (see [28], [29], [30]). Note that J.J. Sylvester has found other formulas for these remainders in [43] – although their validity has been established only very recently, see [31].

2. Using the algorithm in 1. and passing to the limit, Wroński [15] (see also [20]) also solved the following important *factorization problem*:

Suppose that we are given a normed polynomial $W(x) \in \mathbb{C}[x]$*, which does not have roots of absolute value* 1*. Let*

$$A := \{a \in \mathbb{C} : W(a) = 0,\, |a| > 1\}, \qquad B := \{b \in \mathbb{C} : W(b) = 0,\, |b| < 1\}.$$

Extract a factor $\prod_{b \in B}(x - b)$ *from* $W(x)$.

We give – following Lascoux [29] – Wroński's solution in terms of the *Schur functions* (here we use the definitions and notation for the Schur functions from [29] and [30]). The coefficients of the polynomial $W(x)$, from which we wish to extract a factor corresponding to roots of absolute value smaller than 1, are the elementary symmetric functions of $A \cup B$ – the sum of (multi)sets A and B. Therefore the problem boils down to expressing elementary symmetric functions of the variable B, in terms of the Schur functions of $A \cup B$, denoted by $S_J(A+B)$. Let the cardinality of the (multi)set A be equal to m. For $I \in \mathbb{N}^m$ i $k, p \in \mathbb{N}$ we define

$$I(k) := (i_1 + k, \ldots, i_m + k), \qquad 1^p I(k) := (1, \ldots, 1, i_1 + k, \ldots, i_m + k)$$

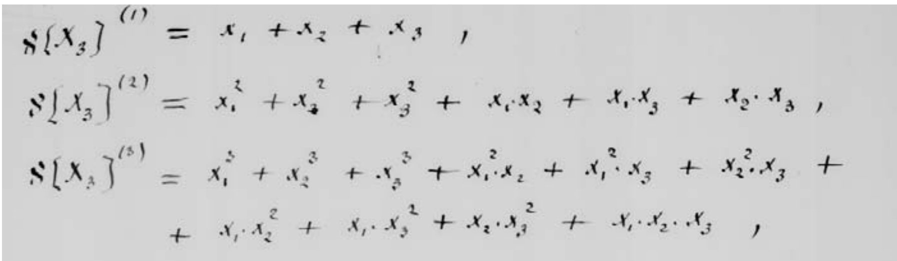
(where 1 is present p times). Let the cardinality of the (multi)set B be equal n. Wroński's theorem (in Lascoux's interpretation [29]) states that

$$\prod_{b \in B}(x - b) = \lim_{k \to \infty} \left(\sum_{0 \leq p \leq n} (-1)^p x^{n-p} \frac{S_{1^p I(k)}(A+B)}{S_{I(k)}(A+B)} \right)$$

(here I is an arbitrary sequence in $\mathbb{N}^m$). Notice that the solution uses a passage to the limit; therefore besides algebraic arguments, transcendental arguments are also used. One can find the proof of this result in [29]. Therefore, we see that Wroński, looking for roots of algebraic equation, *did not* limit himself to using radicals.

3. Assuming that $\deg(F) = \deg(G)+1$, Wroński also found interesting formulas for the remainders $R_i(x)$ in terms of *continued fractions* (see [30], where his formulas are also expressed in terms of the Schur functions).

We note that Wroński also used symmetric functions of the variables $x_1, x_2, \ldots$, and especially *the aleph functions*.



$$\aleph[X_3]^{(1)} = x_1 + x_2 + x_3 \,,$$
$$\aleph[X_3]^{(2)} = x_1^2 + x_2^2 + x_3^2 + x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 \,,$$
$$\aleph[X_3]^{(3)} = x_1^3 + x_2^3 + x_3^3 + x_1^2 \cdot x_2 + x_1^2 \cdot x_3 + x_2^2 \cdot x_3 + x_1 \cdot x_2^2 + x_1 \cdot x_3^2 + x_2 \cdot x_3^2 + x_1 \cdot x_2 \cdot x_3 \,,$$

The aleph functions in three variables of degree 1, 2 and 3
from Wroński's manuscript

More generally, for $n \in \mathbb{N}$ we let $X_n = \{x_1, \ldots, x_n\}$ and define functions $\aleph[X_n]^i$ by the formula

$$\sum_{i \geq 0} \aleph[X_n]^i = \prod_{j=1}^{n} (1 - x_j)^{-1} \,,$$

i.e., $\aleph[X_n]^i$ is the sum of all monomials of degree i. Wroński considered these functions as "more important" than the "popular" *elementary* symmetric functions. This intuition of Wroński has gained – let's call it – justification in the theory of *symmetrization operators* [30] – in the theory of Gröbner bases – so important in computer algebra (see, e.g., [39]), as well as in the modern *intersection theory* in algebraic geometry [11], using rather *Segre classes*, which correspond to aleph functions, than *Chern classes*, corresponding to elementary symmetric polynomials. Here we quote one of the main creators of intersection theory – W. Fulton [11], p. 47:

Segre classes for normal cones have other remarkable properties not shared by Chern classes.

All this shows that Wroński had an unusually deep intuition regarding mathematics.

In Wroński's time there was a fascination with *continued fractions*[5]. As much as the earlier generations of mathematicians (Bombelli, Cataldi, Wallis, Huygens, Euler, Lambert, Lagrange...) were interested *mainly* in expressing irrational numbers as continued fractions, obtaining such spectacular results as:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}, \qquad e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots}}}},$$

$$\pi = 3 + \cfrac{1}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{\ddots}}}}$$

– in Wroński's time attempts were made *mainly* to express functions of one variable as continued fractions. Already in *Philosophy of Mathematics* [14] from 1811 Wroński considered *the problem of interpolation* of a function of one variable $f(x)$ by continued fractions. Let $g(x)$ be an auxiliary function vanishing at 0, and ξ – an auxiliary parameter. Wroński gives the expansion of $f(x)$ as a continued fraction

$$f(x) = c_0 + \cfrac{g(x)}{c_1 + \cfrac{g(x-\xi)}{c_2 + \cfrac{g(x-2\xi)}{c_3 + \cfrac{g(x-3\xi)}{\ddots}}}}$$

expressing unknown parameters c_0, c_1, c_2, ... in terms of $f(0)$, $f(\xi)$, $f(2\xi)$, ... This is connected with the *Thiele continued fractions* [44]. A few years later Wroński gave even more general continued fractions, considering instead of one auxiliary function $g(x)$ a system of functions $g_0(x), g_1(x), \ldots$, vanishing at various points:

$$0 = g_0(\alpha_0) = g_1(\alpha_1) = g_2(\alpha_2) = \cdots .$$

[5]The history of continued fractions is described in [4]. The nineteenth century can be described – without exaggeration – as the golden age of continued fractions. This was the time when this topic was known to *every* mathematician. The following are among those who were seriously involved: Jacobi, Perron, Hermite, Gauss, Cauchy and Stieltjes. Mathematicians studied continued fractions involving functions as well as those involving numbers (the same remark applies to the previous century, especially regarding the activity of Euler and Lambert). However it was Wroński who was the pioneer of functional continued fractions in interpolation theory – this fact, surprisingly, was noticed for the first time only recently by Lascoux [30].

Wroński gives determinantal formulas $f(\alpha_i)$, $i = 0, 1, \dots$, for the coefficients c_j, $j = 0, 1, \dots$, in the expansion

$$f(x) = c_0 + \cfrac{g_0(x)}{c_1 + \cfrac{g_1(x)}{c_2 + \cfrac{g_2(x)}{c_3 + \cfrac{g_3(x)}{\ddots}}}} .$$

These expansions are connected with the *Stieltjes continued fractions* [42] and play a key role in interpolation theory. In his book [30], Lascoux called them the *Wroński continued fractions*, therefore bringing Wroński's name for the second time (after Wrońskians) into the mathematical literature. More details, as well as specific references to Wroński's papers, can be found in [30].

In 1812 Wroński published *Criticism of Lagrange's theory of analytic functions* [16]. Wroński's views on this subject were shared by a number of other mathematicians, among others Poisson. The criticism regarded particularly the problem of interpretation of "infinitely small values" and the incomplete derivation of the Taylor formula. This is the paper where Wroński introduces for the first time "combinatorial sums" containing derivatives, today called *Wrońskians.*

In these years Wroński searched for a solid foundation for his plans; he thought that he will find it in the most distinguished scientific institution: The French Academy. In 1810 he sent to the Academy – to establish contact – the article *On the fundamental principles of algorithmic methods* containing the "Highest Law", which allows expanding functions of one variable into a series[6]. The committee judging the article had established that Wroński's formula encompasses all expansions known until that time, the Taylor formula for example, but withheld confirming the validity of formula in its most general form. Wroński insisted on a definitive answer, and – in anticipation of a dispute – declined to accept the status of a Corresponding Member of the Academy suggested by Lagrange. The Academy did not give an official response neither to Wroński's reply, nor to his further letters. On top of that, such a serious work as the earlier mentioned *Philosophy of Mathematics* was not noticed by the Academy, as well as the article *On solutions of equations.* Of course, the attitude of the Academy with respect to *Criticism of Lagrange's theory of analytic functions* could not have been different and not hostile towards Wroński. In the committee judging the article was … Lagrange himself and his colleagues. Because of the negative opinion, Wroński withdrew his

[6] The name "The Highest Law" used to describe the possibility of expanding a function into a series may seem a bit pompous. We should remember, however, that mathematicians of that time were fascinated by the possibility of "passing to infinity". This fascination concerned not only infinite series, but also infinite continued fractions. Today there is nothing special about infinity: if a space needs to be compactified, one just "adds a point at infinity"…. Wroński and his contemporaries treated infinity with great owe and respect as a great transcendental secret.

paper from the Academy, directing – according to his character – bitter words towards the academics from Paris (phrases: "born enemies of truth", "les savants sur brevets" are among . . . the milder ones).

At this time, Wroński's material situation has become much worse. While working on his publications, he has neglected his teaching, and the illness of his wife and child forced him to sell all of his possessions. Despite all efforts, the child could not be saved, and Wroński dressed in worn out clothes and clogs. He asked Napoleon himself for funding, however Napoleon was not interested in his activity. Wroński lived on the edge of the large Polish emigration in Paris, even though – as he bitterly states in his diaries – he dedicated his treatise on equations to his Polish homeland.

A (financially) important moment was Wroński's meeting with P. Arson, a wealthy merchant and banker from Nice, to whom Wroński was introduced by his old friend Ph. Girard (by the way, Girard was the founder of Żyrardów, a Polish town). Arson, fascinated by Wroński's ideas, promised to fund his activity for a few years. In return, Wroński was to reveal him the secret of the Absolute. This strange bond of a philosopher and a banker lasted until 1816 r. Arson, Wroński's secretary, finally insisted the revealing of the secret, and when the mentor did not do so, Arson took him to court. The matter become so well known, that after a few years it was the theme of one of Balzac's books *The search for the Absolute*. Arson resigned his post, but had to pay the debts of his ex-mentor (because Wroński won in court, by convincing the judge, that he knows the mystery of the Absolute). At that time Wroński publishes *Le Sphinx*, a journal which was to popularize his social doctrines.

The years 1814–1819 bring more Wroński's publications, mostly in the area of philosophy of mathematics: *Philosophy of infinity* (1814), *Philosophy of algorithmic techniques* (1815, 1816, 1817), *Criticism of Laplace's generating functions*. The academy has neglected all these publications.

5. The stay in England. In 1820 Wroński went to England, in order to compete for an award in a contest for a method to measure distances in navigation. This trip was very unfortunate. On the boarder, the customs officials took possession of all his instruments, which Wroński never recovered. His papers were regarded as theoretical, and as such not suitable for the award. Finally, the secretary of the Board of Longitude, T. Young has made certain important modifications in the tables of his own authorship on the basis of Wroński's notes sent to him, "forgetting" to mention who should be given credit for these improvements. Of course, Wroński protested by sending a series of letters, also to the Royal Society. He had never received a response.

The very original *Introduction to the lectures of mathematics* [18] dates from this period (see also [22]), written in English and published in London in 1821. Wroński states there, that all positive knowledge is based on mathematics or in some sense draws from it. Wroński divides the development of mathematics into $4 + 1$ periods:

1. works of the scholars of East and Egypt: concrete mathematics was practiced, without the ability to raise to abstract concepts;
2. the period from Tales and Pythagoras until the Renaissance: the human mind rose to the level of high abstraction, however the discovered mathematical truths existed as unrelated facts, not connected by a general principle as, e.g., the description of the properties of conic sections;
3. the activity of Tartaglia, Cardano, Ferrari, Cavalieri, Bombelli, Fermat, Vieta, Descartes, Kepler,. . . : mathematics rose to the study of general laws thanks to algebra, but the achievements of mathematics are still "individual" – the "general" laws of mathematics were still unknown;
4. the discovery of differential and integral calculus by Newton and Leibniz, expansion of functions into series, continued fractions popularized by Euler, generating functions of Laplace, theory of analytic functions of Lagrange. The human mind was able to raise from the consideration of quantities themselves to the consideration of their creation in the calculus of functions, i.e., differential calculus.

The fifth period should begin with the discovery of the Highest Law and algorithmic techniques by Wroński; the development of mathematics should be based on the most general principles – "absolute ones" – encompassing all of mathematics. This is because all the methods and theories up to that time do not exhaust the essence of mathematics, as they lack a general foundation, from which everything would follow. They are relative, even though science should look for absolute principles. Therefore, the fifth period foresees a generalization of mathematics. Indeed, this will happen later, but not on the basis of philosophy, as Wroński wanted. We mention here the following mathematical theories, which appeared soon: group theory (Galois), projective geometry (Monge, Poncelet), noneuclidean geometries (Łobaczewski, Bolyai, Gauss, Riemann) and set theory (Cantor).

6. Canons of logarithms – a bestseller. In 1823 Wroński is back in Paris and is working on mathematical tables and construction of mathematical instruments: an arithmetic ring (for multiplication and division) and "arithmoscope" (for various arithmetic operations). Among Wroński's achievements in this matter, is his *Canon of logarithms* [19] (see also [23]). With the help of appropriate logarithms and cleverly devised decomposition of a number into certain parts, common for different numbers, he was able to set these parts in such a way, that these tables, even for very large numbers, fit onto one page. For logarithms with 4 decimal places the whole table can be fitted into a pocket notebook. Wroński's *Canon of logarithms* has been published many times in different languages (and shows that, besides very hard to read treatises, he could also produce works which are easier to comprehend).

In 1826 Wroński went to Belgium for a short time, where he was able to interest Belgian mathematicians in his achievements. In fact Belgian scientists were the first to bring Hoene-Wroński into worldwide scientific literature.

In 1829 Wroński, fascinated by the advances in technology, published a treatise on the steam engine.

7. Letters to the rulers of Europe. From around 1830 until the end of his life Wroński focused exclusively on the notion of messianism. At that time he published his well-known *Address to the Slavonic nations about the destiny of the World* and his most well-known works: *Messianism*, *Deliberations on messianism* and *Introduction to messianism.* At that time he also sent memoranda to Pope Leon XII and the Tsars, so that they would back his messianistic concept.

One should also mention, that Wroński sent letters to the rulers of Europe instructing them how they should govern. These letters contained specific mathematical formulas, how to rule. Here is an example of such a formula from *The Secret letter to his Majesty Prince Louis-Napoléon* [21] from 1851.

Let a be the degree of anarchy, d – the degree of despotism. Then

$$a = \left(\frac{m+n}{m} \cdot \frac{m+n}{n}\right)^{p-r} \cdot \left(\frac{m}{n}\right)^{p+r} = \left(\frac{m+n}{n}\right)^{2p} \cdot \left(\frac{m}{m+n}\right)^{2r},$$

$$d = \left(\frac{m+n}{m} \cdot \frac{m+n}{n}\right)^{r-p} \cdot \left(\frac{n}{m}\right)^{p+r} = \left(\frac{n}{m+n}\right)^{2p} \cdot \left(\frac{m+n}{m}\right)^{2r},$$

where m = number of members of the liberal party, p = the deviation of the philosophy of the liberal party from true religion, n = the number of members of the religious party, r = the deviation of the religious party from true philosophy. According to Wroński, for France one should take $p = r = 1$, and then

$$a = \left(\frac{m}{n}\right)^2, \qquad d = \left(\frac{n}{m}\right)^2.$$

Moreover, $\frac{m}{n} = 2$, and so $a = 4$, $d = \frac{1}{4}$. This means that, political freedom – in France of Wroński's time – is four times the normal one, and the authority of the government is one quarter of what is essential.

(The application of the above formulas to the current Polish political reality would be interesting. . . .)

8. Philosophy. I. Kant's philosophy was the starting point of the philosophy of Wroński, who has transformed it into metaphysics in a way analogous to Hegel's approach. Wroński has not only created a philosophical system, but also its applications to politics, history, economy, law, psychology, music (see [38]) and education. Existence and knowledge followed from the Absolute, which he understood either as God, or as the spirit, wisdom, a thing in itself. He did not describe it, but he tried to infer from it a universal law, which he called "The Law of Creation".

In his philosophy of history he predicted reconstruction of the political system, from one full of contradictions to a completely reasonable one. In the history of philosophy he distinguished four periods, each of which imposed on itself different aims:

1. east – material aims;
2. Greak-Roman – moral aims;

3. medieval – religious aims;
4. modern, until the XVIII century – intellectual aims.

He treated the XIXth century as a transitional period, a time of competition of two blocs: conservative bloc whose aim is goodness and liberal bloc whose aim is the truth.

Wroński is the most distinguished Polish messianic philosopher. It is him (and not Mickiewicz nor Towiański) who introduced the notion of "messianism". Wroński held, that it is the vocation of the human race to establish a political system based on reason, in which the union of goodness and truth and religion and science will take place. The Messiah, who will bring the human race into the period of happiness, is – according to Wroński's concepts – precisely *philosophy.*

Jerzy Braun was an expert and promoter of Wroński's philosophy in Poland. His article *Aperçu de la philosophie de Wroński* published in 1967 is much valued by the French scholars of Wroński's philosophy.

9. Mathematics: The Highest Law, Wrońskians. Essentially, Wroński worked on mathematical analysis and algebra. We have already discussed Wroński's contributions to algebra. In analysis[7] he was especially interested in expanding functions in a *power series* and *differential equations.* Wroński's most interesting mathematical idea was his general method of expanding a function $f(x)$ of one variable x into a series

$$f(x) = c_1 g_1(x) + c_2 g_2(x) + c_3 g_3(x) + \cdots ,$$

when the sequence of functions $g_1(x), g_2(x), \ldots$ is given beforehand, and $c_1, c_2, \ldots$ are numerical coefficients to be determined. Notice that if

$$g_1(x), g_2(x), \ldots$$

form an orthonormal basis with respect to the standard, or any other, inner product $(\cdot\,,\,\cdot)$ on the (infinite-dimensional) vector space of polynomials of one variable, then for each i we have

$$c_i = \big(f(x), g_i(x)\big).$$

However, such a simple situation rarely happens. Wroński gave his method of finding the coefficients c_i the rank of *The Highest Law.* From today's point of view the method lacked precision and rigor (for example, Wroński did not consider the matter of convergence), however it contained – besides interesting calculations – useful ideas. These ideas were used much later by Stefan Banach, who formulated them precisely and enriched them with topological concepts, and proved that the Highest Law of Hoene-Wroński can be used in what is called today a *Banach space*, as well as in the theory of *orthogonal polynomials.* I will mention here a little known letter of Hugo Steinhaus to Zofia Pawlikowska-Brożek:

[7] Strictly speaking, making a distinction between algebra and analysis is not strictly correct, since Wroński often mixed algebraic and analytic methods.