Branko Grünbaum

Convex Polytopes

Second Edition

First Edition Prepared
with the Cooperation of
Victor Klee, Micha Perles, and Geoffrey C. Shephard

Second Edition Prepared by
Volker Kaibel, Victor Klee, and Günter M. Ziegler

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In humble homage
dedicated to the memory of the outstanding geometer

Ernst Steinitz
(1871–1928)

PREFACE

Convex polytopes—as exemplified by convex polygons and some three-dimensional solids—have been with us since antiquity. However, hardly any results worth mentioning and dealing specifically with the *combinatorial properties* of convex polytopes were discovered prior to Euler's famous theorem concerning the number of vertices, edges, and faces of three-dimensional polytopes. Euler's relation, hailed by Klee as "the first landmark" in the theory of convex polytopes, served as the starting point of a multitude of investigations which led to the determination of its limits of validity, and helped focus attention on the notion of convexity. Additional ideas and results came from such mathematicians as Cauchy, Steiner, Sylvester, Cayley, Möbius, Kirkman, Schläfli, and Tait. Since the middle of the last century, polytopes of four or more dimensions attracted interest; crystallography, generalizations of Euler's theorem, the search for polytopes exhibiting regularity features, and many other fields provided added impetus to the investigation of convex polytopes.

About the turn of the century, however, a steep decline in the interest in convex polytopes was produced by two causes working in the same direction. Efforts at enumerating the different combinatorial types of polytopes, started by Euler and pursued with much patience and ingenuity during the second half of the XIXth century, failed to produce any significant results even in the three-dimensional case; this lead to a widespread feeling that the interesting problems concerning polytopes are hopelessly hard. Simultaneously, the ascendance of Klein's "Erlanger Program" and the spread of its normative influence tended to cast the preoccupation with the combinatorial theory of convex polytopes into a rather disreputable rôle—and that at a time when such "legitimate" fields as algebraic geometry and in particular topology started their spectacular development.

Due to this combination of circumstances and pressures it is probably not too surprising that only few specialized directions of research in polytopes remained active during the first half of the present century. Stretching slightly the time limits, the most prominent examples of those efforts were: Minkowski's fundamental contributions, related to his work on convexity in general, and applications to number theory in particular; Coxeter's work on regular polytopes; A. D. Aleksandrov's investigations in the metric theory of polytopes.

Nevertheless, as far as "main-line mathematics" is concerned, the combinatorial theory of convex polytopes was "out". Despite the appreciable number of published papers dealing with isolated (mostly extremal) problems, the whole area was relegated to the borderline between serious research and amateurish curiosity. The one notable exception in this respect among first-rank mathematicians was Ernst Steinitz, who devoted a sizable part of his life and efforts to the combinatorial theory of polytopes. Unfortunately, his beautiful results did not become as well known as they deserve, and till very recently did not stimulate additional research.

It was mainly under the influence of computational techniques (in particular, linear programming) that a renewed interest in the combinatorial theory of convex polytopes became evident slightly more than ten years ago. The phenomenon of "neighborly polytopes" was rediscovered by Gale in 1955 (the rather involved history of this concept is related in Section 7.4). Neighborly polytopes, and Motzkin's "upper-bound conjecture" (1957) served as focal points for many investigations (see Chapters 9 and 10). Despite many scattered results on the upper-bound conjecture and other combinatorial problems about convex polytopes, obtained by different authors in the first few years of the 1960's, the emergence of a theory proper began only with Klee's work, starting in 1962. Klee's results on the Dehn–Sommerville equations (the interesting history of this topic is given in Section 9.8) and his almost complete solution of the upper-bound conjecture were the source and basis for many of the subsequent developments.

During the last three years, research into the combinatorial structure of convex polytopes has grown at an astonishing rate. It would be premature to attempt to give here even the briefest historic outline of this period. Instead, detailed bibliographic references are supplied with each topic discussed in the book.

The present book grew out of lecture notes prepared by the author for a course on the combinatorial theory of convex polytopes given at the Hebrew University of Jerusalem in 1964/65. The main part of the final version was written while the author was lecturing on the same topic at the Michigan State University in East Lansing during 1965/66. The various parts of the book may be described briefly as follows:

The first four chapters are introductory and are meant to acquaint the reader with some basic facts on convex sets in general, and polytopes in particular; as well as to provide "experimental material" in the form of examples.

Some basic tools for the investigation of polytopes are described in detail in Chapter 5; most of them are used in different subsequent sections. In Chapters 6 and 7 some of those techniques are applied to polytopes with "few" vertices, and to neighborly polytopes.

Chapters 8, 9, and 10 have as common topic the relations between the numbers of faces of different dimensions. Starting with Euler's equation, the Dehn–Sommerville equations for simplicial polytopes (and for certain other families) are discussed and used in the (partial) solution of the upper-bound conjecture. Chapter 14 is related to Chapter 9 by the similarity of the equations involved.

Chapters 11 and 12 deal with problems of a more topological flavor, while Chapter 13 discusses the much more detailed results known about 3-dimensional convex polytopes.

Chapter 15 contains a survey of the known results concerning the representation of polytopes as sums of other polytopes.

A summary of the available results about graphs of polytopes and paths in those graphs, as well as their relation to various problems that arose in applications, forms the topic of Chapters 16 and 17.

Chapter 18 deals with a topic related to convex polytopes more by the spirit of the problems considered than by actual interdependence: partitions of the (projective) space by hyperplanes.

In the last chapter a number of unrelated areas is surveyed. Their inclusion—at the expense of other topics which could have been included—is due to the author's interest in them.

It is hoped that parts of the book will prove suitable as texts for a number of different courses. On the other hand, the book is meant to serve as a ready reference for research workers; hence an attempt at completeness was made both in the coverage of the topics discussed, and in the bibliography. While the author is confident that the current surge of interest and research in the combinatorial properties of convex polytopes will continue and will render the book obsolete within a few years, he may only hope that the book itself will contribute to the revitalization of the field and act as a stimulant to further research. (Some of the results that came to the author's attention after completion of the manuscript in August 1966 are mentioned in the Addendum on pp. 426–428.)

It was the author's good fortune to obtain the cooperation of his friends and colleagues Victor Klee, M. A. Perles, and G. C. Shephard. Professor Klee wrote Chapters 16 and 17, while Professor Shephard contributed Chapter 15, Section 14.3, and part of Section 14.4. Professor Perles permitted the inclusion of many of his unpublished results; they are reproduced in Sections 5.1, 5.4, 5.5, 6.3, 11.1, 12.3, and in many other places throughout the book. In addition, Perles corrected many of the errors contained in the various preliminary versions, and contributed a large number of exercises. The author's indebtedness to Klee, Perles, and Shephard, hardly needs elaboration.

Thanks are also due to many other colleagues who contributed to the effort through discussions, suggestions, corrections etc. It would not be feasible to mention them all here. Particular thanks are due to W. E. Bonnice, L. M. Kelly, J. R. Reay, V. P. Sreedharan, and B. M. Stewart, all colleagues at Michigan State University during 1965/66, whose patience, encouragement and help during the most exasperating stages are gladly acknowledged and deeply appreciated.

The author gratefully acknowledges the financial support obtained at various times from the National Science Foundation and from the Air Force Office of Scientific Research, U.S. Air Force. Much of the research that is being published for the first time in the present book was conducted under the sponsorship of those agencies. Professor Klee acknowledges some helpful suggestions from David Barnette, and financial support from the University of Washington, the National Science Foundation, the RAND Corporation, and especially from the Boeing Scientific Research Laboratories; Chapters 16 and 17 appeared in a slightly different form as a BSRL Report.

The author's most particular thanks go to his wife Zdenka; without her encouragement and patience the book would have never been completed.

University of Washington, Seattle
December 31, 1966 BRANKO GRÜNBAUM

Preface to the 2002 edition

There is no such thing as an "updated classic"—so this is not what you have in hand.

In his 1966 preface, Branko Grünbaum expressed confidence "that the current surge of interest and research in the combinatorial properties of convex polytopes will continue and will render the book obsolete in a few years." He also stated his "hope that the book itself will contribute to the revitalization of the field and act as a stimulant to further research."

This hope has been realized. The combinatorial study of convex polytopes is today an extremely active and healthy area of mathematical research, and the number and depth of its relationships to other parts of mathematics have grown astonishingly. To some extent, Branko's confidence in the obsolescence of his book was also justified, for some of the most important open problems mentioned in it have by now been solved. However, the book is still an outstanding compendium of interesting and useful information about convex polytopes, containing many facts not found elsewhere.

Major topics, from Gale diagrams to cubical polytopes, have their beginnings in this book. The book is comprehensive in a sense that was never achieved (or even attempted) again. So it is still a major reference for polytope theory (without needing any changes).

Unfortunately, the book went out of print as early as 1970, and some of our colleagues have been looking for "their own copy" since then. Thus, responding to "popular demand", there have been continued efforts to make the book accessible again. Now we are happy to say: *Here it is!*

The present new edition contains the full text of the original, in the original typesetting, and with the original page numbering—except for the table of contents and the index, which have been expanded. You will see yourself all that has been added: The notes that we provide are meant to help to bridge the thirty-five years of intensive research on polytopes that were to a large extent initiated, guided, motivated, and fuelled by this book. However, to make this edition feasible, we had to restrict these notes severely, and there is no claim or even attempt for any complete coverage. The notes that we provide for the individual chapters try to summarize a few important developments with respect to the topics treated by Grünbaum, quite a remarkable number of them triggered by his exposition. Nevertheless, the selection of topics for these notes is clearly biased by our own interests.

The material that we have added provides a direct guide to more than 400 papers and books that have appeared since 1967; thus references like "Grünbaum [a]" refer to the additional bibliography which starts on page 448a. Many of those publications are themselves surveys, so there is also much work to which the reader is guided indirectly. However, there remain many gaps that we would have liked to fill if space permitted, and we apologize to fellow researchers whose favorite polytopal papers are not mentioned here.

Principal references to "polytope theory since Grünbaum's book" that we have relied on include the books by McMullen–Shephard [b], Brøndsted [a], Yemelichev–Kovalev–Kravtsov [a], Ziegler [a], and Ewald [a], as well as the surveys by Grünbaum–Shephard [a], Grünbaum [d], Bayer–Lee [a], and Klee–Kleinschmidt [b]. Furthermore, we want to direct the readers' attention to Croft–Falconer–Guy [a] for (more) unsolved problems about polytopes.

We have taken advantage of some tools available in 2002 (but not in 1967), in order to compute and to visualize examples. In particular, the figures that appear in the additional notes were computed in the `polymake` framework by Gawrilow–Joswig [a, b], and were visualized using `javaview` by Polthier et al. [a].

Moreover and most of all, we are indebted to a great number of very helpful and supportive colleagues—among them Marge Bayer, Lou Billera, Anders Björner, David Bremner, Christoph Eyrich (LaTeX with style!), Branko Grünbaum, Torsten Heldmann, Martin Henk, Michael Joswig, Gil Kalai, Peter Kleinschmidt, Horst Martini, Jirka Matoušek, Peter McMullen, Micha Perles, Julian Pfeifle, Elke Pose, Thilo Schröder, Egon Schulte, and Richard Stanley—who have provided information and assistance on the way to completion of this long-planned "Grünbaum reissue" project.

Berlin/Seattle, September 2002,

Volker Kaibel · Victor Klee · Günter M. Ziegler

CONTENTS

CHAPTER 1

Notation and Prerequisites

1.1 Algebra

With few exceptions, we shall be concerned with convexity in R^d, the d-dimensional real Euclidean space. Lower case characters such as a, b, x, y, z shall denote points of R^d, as well as the corresponding vectors; 0 is the origin as well as the number zero. Capitals like A, B, C, K shall denote sets; occasionally single points, if considered as one-pointed sets, shall be denoted by capitals. Greek characters α, β, λ, μ, etc., shall denote reals, while n, k, i, j shall be used for integers. The coordinate representation of a point $a \in R^d$ shall be $a = (\alpha_i) = (\alpha_1, \alpha_2, \cdots, \alpha_d)$.

Sets defined explicitly by specifying their elements will be written in the forms $\{a_1, \cdots, a_k\}$, $\{a_1, \cdots, a_n, \cdots\}$, or $\{a \in A \mid a \text{ has property } \mathscr{S}\}$, the last expression indicating all those elements of a set A which have a certain property $\mathscr{S}$. Finite or infinite *sequences* (of not necessarily different elements) will be denoted by $(a_1, \cdots, a_k)$ or $(a_1, \cdots, a_n, \cdots)$; the first expression will also be called a *k-tuple*. For the set-theoretic notions of *union*, *intersection*, *difference*, *subset* we shall use the symbols $\cup$, $\cap$, $\sim$, and $\subset$. The empty set will be denoted by $\emptyset$, while card A will denote the cardinality of the set A.

The algebraic signs are reserved for algebraic operations; thus

$$a \pm b = (\alpha_i) \pm (\beta_i) = (\alpha_i \pm \beta_1)$$

$$\lambda a = \lambda(\alpha_i) = (\lambda \alpha_i)$$

$$A \pm B = \{a \pm b \mid a \in A, b \in B\}$$

$$\lambda A = \{\lambda a \mid a \in A\}.$$

If a set A consists of a single point a we shall use the simplified notation $a + B$ instead of $\{a\} + B = A + B$. The set $(-1)A$ will be denoted $-A$. The set $x + \lambda B$, for $\lambda \neq 0$, is said to be *homothetic* to B, and *positively homothetic* if $\lambda > 0$.

The *scalar product* $\langle a, b\rangle$ of vectors $a, b \in R^d$ is the real number defined by

$$\langle a, b\rangle = \sum_{i=1}^{d} \alpha_i \beta_i .$$

The most important properties of the scalar product are

$$\langle a, b\rangle = \langle b, a\rangle$$
$$\langle \lambda a, b\rangle = \lambda \langle a, b\rangle$$
$$\langle a + b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$$
$$\langle a, a\rangle \geq 0 \text{ with equality if and only if } a = 0 .$$

If $\langle a, b\rangle = 0$ then a and b are said to be *orthogonal* to each other. If $\langle a, a\rangle = 1$ then a is called a *unit vector*. In the sequel, the letter u (with or without indices) shall be used *only* for unit vectors.

A *hyperplane* H is a set which may be defined as $H = \{x \in R^d \mid \langle x, y\rangle = \alpha\}$ for suitable $y \in R^d$, $y \neq 0$, and α. An *open halfspace* [*closed halfspace*] is defined as $\{x \in R^d \mid \langle x, y\rangle > \alpha\}$ [respectively $\{x \in R^d \mid \langle x, y\rangle \geq \alpha\}$] for suitable $y \in R^d$, $y \neq 0$, and α. Clearly, $\{x \in R^d \mid \langle x, y\rangle < \alpha\}$ is also an open halfspace for $y \neq 0$; similarly for closed halfspaces.

Each hyperplane has a translate which is (isomorphic to) a $(d - 1)$-dimensional Euclidean space R^{d-1}. For each hyperplane H which does not contain the origin 0 there exists a unique representation $H = \{x \in R^d \mid \langle x, u\rangle = \alpha\}$ in which u is a unit vector and $\alpha > 0$.

If $x, x_i \in R^d$, we shall say that x is a *linear combination* of the x_i's provided

$$x = \sum_{i=1}^{k} \lambda_i x_i$$

for suitable real numbers λ_i.

If $x = \sum_{i=1}^{k} \lambda_i x_i$ for reals λ_i satisfying $\sum_{i=1}^{k} \lambda_i = 1$ we shall say that x is an *affine combination* of the x_i's.

A set $X = \{x_1, \cdots, x_k\}$ is *linearly* [respectively *affinely*] *dependent* provided 0 is representable as a linear combination $0 = \sum_{i=1}^{k} \lambda_i x_i$ in which some $\lambda_i \neq 0$ [and $\sum_{i=1}^{k} \lambda_i = 0$]. If a set X fails to be linearly [affinely] dependent we call it *linearly* [*affinely*] *independent*. In any linearly [affinely] dependent set some point is a linear [affine] combination of the remaining points. The d-dimensional space contains d-membered sets which are linearly independent, but every $(d + 1)$-membered set in R^d is linearly dependent. A set $X = \{x_0, x_1, \cdots, x_k\}$ is affinely dependent [independent] if and only if the set $(X \sim \{x_0\}) - x_0 = \{x_1 - x_0, x_2 - x_0, \cdots, x_k - x_0\}$ is linearly dependent [independent]. For any $x \in R^d$ the sets X and $x + X$ are simultaneously affinely dependent or independent.

The set of all affine combinations of two different points $x, y \in R^d$ is the *line* $L(x, y) = \{(1 - \lambda)x + \lambda y | \lambda \text{ real}\}$. If $x', y' \in L(x, y)$ and $x' \neq y'$ then $L(x', y') = L(x, y)$.

If a set H has the property that $L(x, y) \subset H$ whenever $x, y \in H, x \neq y$, we call H a *flat* (or an *affine variety*). Clearly, the set of all affine [linear] combinations formed from all finite subsets of a given set A is a flat [subspace]; it is denoted by aff A [lin A] and is called the *affine hull* [*linear hull*] of A. The family of all flats in R^d contains R^d, $\emptyset$, all one-pointed sets, all lines, all hyperplanes; also, it is *intersectional*: if all H_α's are flats, so is $\bigcap_\alpha H_\alpha$. The affine hull aff A of a set A may equivalently be defined as the intersection of all flats which contain A. Similar statements hold for linear hulls. The formation of the affine hull is *translation invariant*, i.e. $\text{aff}(x + A) = x + \text{aff} A$.

Every nonempty flat H is a translate $H = x + V$ of some subspace V of R^d, and is therefore isomorphic to a Euclidean space of a certain dimension $r \leq d$; the *dimension* of H (and of V) is then $r = \dim H = \dim V$. A flat of dimension r will be called an r-flat. We agree to put $\dim \emptyset = -1$. In general, instead of saying 'an object of dimension r' we shall use the shorter term 'r-object'; for example, d-space, r-subspace, etc. If A is any subset of R^d, its *dimension* dim A is defined by $\dim A = \dim \text{aff} A$.

Each r-flat contains $r + 1$ affinely independent points, but each $(r + 2)$-membered set of its points is affinely dependent.

If $A = \{a_1, a_2, \cdots, a_k\}$, where $a_i = \{\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{id}\}$, then the maximal number of linearly independent members of A equals the rank of the matrix

$$\begin{pmatrix} \alpha_{11} \, \alpha_{12} \cdots \alpha_{1d} \\ \alpha_{21} \, \alpha_{22} \cdots \alpha_{2d} \\ \cdots\cdots\cdots\cdots \\ \alpha_{k1} \, \alpha_{k2} \cdots \alpha_{kd} \end{pmatrix}$$

while the maximal number of affinely independent members of A equals the rank of the matrix

$$\begin{pmatrix} 1 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1d} \\ 1 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2d} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{kd} \end{pmatrix}.$$

A finite set $X \subset R^d$ is said to be in *general position* provided each subset of X containing at most $d + 1$ members is affinely independent.

The following remark is sometimes useful: Given positive integers d and k, there exists an integer $n(d, k)$ with the property that whenever $A \subset R^d$ satisfies card $A \geq n(d, k)$, there exists a subset B of A such that card $B = k$ and the points of B are in general position in aff B.

Let a transformation T from R^d to R^d be defined by

$$Tx = \frac{Ax + b}{\langle c, x\rangle + \delta},$$

where A is a linear transformation of R^d into itself, b and c are d-dimensional vectors, and δ is a real number, at least one of c and δ being different from 0. Any transformation of this type is called a *projective transformation** from R^d into R^d. Note that T is not defined for x in $N(T) = \{y \mid \langle c, y\rangle + \delta = 0\}$. The set $N(T)$ may be empty (in which case T is an affine transformation); if A is regular and $c \neq 0$, $N(T)$ is a hyperplane (which, in the *projective space*, is mapped by T into the 'hyperplane at infinity'). The reader is invited to verify that collinear points are mapped by projective transformations onto collinear points. A projective transformation T is *nonsingular* provided the matrix $\begin{pmatrix} A' & b' \\ c & \delta \end{pmatrix}$ is regular (here A' is the matrix of A, and b' the transposed vector b); in this case T has an inverse which is again a projective transformation. If $(x_0, \cdots, x_{d+1})$ and $(y_0, \cdots, y_{d+1})$ are two $(d + 2)$-tuples of points in general position in R^d, there exists a unique projective transformation T such that $Tx_i = y_i$ for $i = 0, \cdots, d + 1$; moreover, this T is nonsingular. If K is a subset of R^d, T is said to be *permissible* for K provided $K \cap N(T) = \emptyset$. If $K_i \subset R^d$

* In case of need, the reader should consult a suitable textbook on projective geometry. However, he should bear in mind that we are dealing with Euclidean (or affine) spaces, and nonhomogeneous coordinates, while the most natural setting for projective transformations are projective spaces and homogeneous coordinates.