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Operator Theoretic Aspects of Ergodic Theory

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To our families and friends

Preface

> The best way to start writing, perhaps the only way, is to write on the spiral plan. According to the spiral plan the chapters get written and rewritten in the order 1, 2, 1, 2, 3, 1, 2, 3, 4, etc. […]
>
> Paul R. Halmos [1]

Ergodic theory has its roots in Maxwell's and Boltzmann's kinetic theory of gases and was born as a mathematical theory around 1930 by the groundbreaking works of von Neumann and Birkhoff. In the 1970s, Furstenberg showed how to translate questions in combinatorial number theory into ergodic theory. This inspired a new line of research, which ultimately led to stunning recent results of Host and Kra, Green and Tao, and many others.

In its 80 years of existence, ergodic theory has developed into a highly sophisticated field that builds heavily on methods and results from many other mathematical disciplines, e.g., probability theory, combinatorics, group theory, topology and set theory, even mathematical logic. Right from the beginning, also *operator theory* (which for the sake of simplicity we use synonymously to "functional analysis" here) played a central role. To wit, the main protagonist of the seminal papers of von Neumann (1932b) and Birkhoff (1931) is the so-called *Koopman operator*

$$T : \mathrm{L}^2(\mathrm{X}) \to \mathrm{L}^2(\mathrm{X}) \qquad (Tf)(x) := f(\varphi(x)), \tag{1}$$

where $\mathrm{X} = (X, \Sigma, \mu)$ is the underlying probability space, and $\varphi : X \to X$ is the measure-preserving dynamics. (See Chapter 1 for more details.) By passing from the state space dynamics φ to the (linear) operator T, one linearizes the situation and can then employ all the methods and tools from operator theory, e.g., spectral theory, duality theory, harmonic analysis.

However, this is not a one-way street. Results and problems from ergodic theory, once formulated in operator theoretic terms, tend to emancipate from their parental home and to lead their own life in functional analysis, with sometimes stunning applicability (like the mean ergodic theorem, see Chapter 8). We, as functional analysts, are fascinated by this interplay, and the present book is the result of this fascination.

Scope

The present text can be regarded as a systematic introduction into classical ergodic theory with a special focus on (some of) its operator theoretic aspects; or, alternatively, as a book on topics in functional analysis with a special focus on (some of) their applications in ergodic theory.

Accordingly, its classroom use can be at least twofold. As no prior encounter with ergodic theory is expected, the book can serve as a basis for an introductory course on that subject, especially for students or researchers with an interest in functional analysis. Secondly, as the functional analytic notions and results are often developed here beyond their immediate connection with ergodic theory, the book can also be a starting point for some advanced or “special topics” course on functional analysis with a special view on applications to ergodic theory.

Apart from the classroom use, however, we intend this book as an invitation for anyone working in ergodic theory to learn more about the many operator theoretic aspects of his/her own discipline. Finally—one great hope of ours—the book may prove valuable as a foundation for future research, leading towards new and yet unknown connections between ergodic and operator theory.

Prerequisites

We certainly require familiarity with basic topology, measure theory, and standard functional analysis, see the Appendices A, B, C. As operator theory on Hilbert spaces is particularly important, we devoted an own appendix (Appendix D) to it. Apart from standard material, it also includes some topics usually missing in elementary functional analysis courses, hence the presentation is relatively detailed there. As a rule, whenever there were doubts about what may be considered “standard,” we included full proofs. This concerns, e.g., the Stone–Weierstraß theorem and the Gelfand–Naimark theorem (Chapter 4), Pontryagin duality (Chapter 14) and the Peter–Weyl theorem (Chapter 15), the Szőkefalvi–Nagy dilation theorem (Appendix D), the Riesz representation theorem (Appendix E), von Neumann’s theorem on the existence of point isomorphisms (Appendix F), the theorems of Eberlein, Grothendieck, and Kreĭn on weak compactness, Ellis’ theorem and the existence of the Haar measure (all in Appendix G).

A Short Synopsis

Chapter 1 entitled “What is Ergodic Theory?” contains a brief and intuitive introduction to the subject, including some remarks on its historical development. The mathematical theory then starts in Chapters 2 and 3 with *topological dynamical systems* . There, we introduce the basic notions (transitivity, minimality, and recurrence) and cover the standard examples, constructions and results, for instance, the Birkhoff recurrence theorem.

Operator theory appears first in Chapter 4 when we introduce, as in (1) above, the *Koopman operator* T on the Banach space $\mathrm{C}(K)$ induced by a topological dynamical system $(K;\varphi)$. After providing some classical results on spaces $\mathrm{C}(K)$ (Urysohn’s lemma, theorems of Tietze and Stone–Weierstraß), we emphasize the Banach algebra structure and give a proof of the classical Gelfand–Naimark theorem. This famous theorem allows to represent each commutative C^*-algebra as a space $\mathrm{C}(K)$ and leads to an identification of Koopman operators as the morphisms of such algebras.

In Chapter 5 we introduce *measure-preserving dynamical systems* and cover standard examples and constructions. In particular, we discuss the correspondence of measures on a compact space K with bounded linear functionals on the Banach space

C(K). (The proof of the central result here, the Riesz representation theorem, is deferred to Appendix E.) The classical topics of recurrence and ergodicity as the most basic properties of measure-preserving systems are discussed in Chapter 6

Subsequently, in Chapter 7 , we turn to the corresponding operator theory. As in the topological case, a measure-preserving map φ on the probability space X induces a *Koopman operator* T on each space $\mathrm{L}^p(\mathrm{X})$ as in (1). While in the topological situation we look at the space C(K) as a Banach algebra and at the Koopman operator as an algebra homomorphism, in the measure theoretic context the corresponding spaces are *Banach lattices* and the Koopman operators are *lattice homomorphisms* . Consequently, we include a short introduction into abstract Banach lattices and their morphisms. Finally, we characterize the ergodicity of a measure-preserving dynamical system by the fixed space or, alternatively, by the irreducibility of the Koopman operator.

After these preparations, we discuss the most central operator theoretic results in ergodic theory, von Neumann's *mean ergodic theorem* (Chapter 8) and Birkhoff's *pointwise ergodic theorem* (Chapter 11). The former is placed in the more general context of *mean ergodic operators* , and in Chapter 10 we discuss this concept for Koopman operators of topological dynamical systems. Here, the classical results of Krylov–Bogoljubov about the existence of invariant measures are proved and the concepts of *unique* and *strict ergodicity* are introduced and exemplified with Furstenberg's theorem on group extensions.

In between the discussion of the ergodic theorems, in Chapter 9 , we introduce the concepts of strongly and weakly *mixing* systems. This topic has again a strong operator theoretic flavor, as the different types of mixing are characterized by different asymptotic behavior of the powers T^n of the Koopman operator as $n \to \infty$. Admittedly, at this stage the results on weakly mixing systems are still somehow incomplete as the relative weak compactness of the orbits of the Koopman operator (on L^p -spaces) is not yet taken into account. The full picture is eventually revealed in Chapter 16 , when this compactness is studied in detail (see below).

Next, in Chapter 12 , we consider different concepts of "isomorphism"—point isomorphism, measure algebra isomorphism, and Markov isomorphism—of measure-preserving systems. From a classical point of view, the notion of point isomorphism appears to be the most natural. In our view, however, the Koopman operators contain all essential information of the dynamical system and underlying state space maps are secondary. Therefore, it becomes natural to embed the class of "concrete" measure-preserving systems into the larger class of "abstract" measure-preserving systems and use the corresponding notion of (Markov) isomorphism. By virtue of the Gelfand–Naimark theorem, each abstract measure-preserving system has many concrete *topological models* . One canonical model, the *Stone representation* is discussed in detail.

In Chapter 13 , we introduce the class of *Markov operators* , which plays a central role in later chapters. Different types of Markov operators (embeddings, factor maps, Markov projections) are discussed and the related concept of a *factor* of a measure-preserving system is introduced.

Compact groups feature prominently as one of the most fundamental examples of dynamical systems. A short yet self-contained introduction to their theory is the topic of Chapter 14 . For a better understanding of dynamical systems, we present the essentials of Pontryagin's duality theory for compact/discrete Abelian groups. This chapter is accompanied by the results in Appendix G, where the existence of the Haar measure and Ellis' theorem for compact semitopological groups is proved in its full generality. In Chapter 15 , we discuss group actions and linear representations of compact groups on Banach spaces, with a special focus on representations by Markov operators.

In Chapter 16 , we start with the study of compact *semigroups* . Then we develop a powerful tool for the study of the asymptotic behavior of semigroup actions on Banach spaces, the *Jacobs–de Leeuw–Glicksberg* (JdLG-) decomposition. Applied to the semigroup generated by a Markov operator T it yields an orthogonal splitting of the corresponding L^2 -space into its "reversible" and the "almost weakly stable" part. The former is the range of a Markov projection and hence a factor, and the operator generates a compact group on it. The latter is characterized by a convergence to 0 (in some sense) of the powers of T .

Applied to the Koopman operator of a measure-preserving system, the reversible part in the JdLG-decomposition is the so-called *Kronecker factor* . It turns out that this factor is trivial if and only if the system is weakly mixing. On the other hand, this factor is the whole system if and only if the Koopman operator has *discrete spectrum* , in which case the system is (Markov) isomorphic to a rotation on a compact monothetic group (Halmos–von Neumann theorem, Chapter 17).

Chapter 18 is devoted to the spectral theory of dynamical systems. Based on a detailed proof of the spectral theorem for normal operators on Hilbert spaces, the concepts of maximal spectral type and spectral multiplicity function are introduced. The chapter concludes with a series of instructive examples.

In Chapter 19 , we approach the Stone–Čech compactification of a (discrete) semigroup via the Gelfand–Naimark theorem and return to topological dynamics by showing some less classical results, like the theorem of Furstenberg and Weiss about multiple recurrence. Here, we encounter the first applications of dynamical systems to combinatorics and prove the theorems of van der Waerden, Gallai, and Hindman.

In Chapter 20 , we describe Furstenberg's correspondence principle, which establishes a relation between ergodic theory and combinatorial number theory. As an application of the JdLG-decomposition, we prove the existence of arithmetic progressions of length 3 in certain subsets of $\mathbb{N}$, i.e., the first nontrivial case of Szemerédi's theorem on arithmetic progressions.

Finally, in Chapter 21 , more ergodic theorems lead the reader to less classical areas and to the front of active research.

What is Not in This Book

Some classical topics of ergodic theory, even with a strong connection to operator theory, have been left out or only briefly touched upon. Our treatment of the *spectral theory* of dynamical systems in Chapter 18 is of an introductory character. From the vast literature on, e.g., spectral realization or spectral isomorphisms of concrete dynamical systems, only a few examples are discussed. For more information on this topic, we refer to Queffélec (1987), Nadkarni (1998b), Lemańczyk (1996), Katok and Thouvenot (2006), Lemańczyk (2009), and to the references therein.

Entropy is briefly mentioned in Chapter 18 in connection with Ornstein's theory of Bernoulli shifts. The reader interested in its theory is referred to, e.g., the following books: Billingsley (1965), Parry (1969a), Ornstein (1974), Sinaĭ (1976), England and Martin (1981), Cornfeld et al. (1982), Petersen (1989), Downarowicz (2011).

Applications of ergodic theory in *number theory* and *combinatorics* are discussed at several places throughout the book, most notably in Chapter 20 where we prove Roth's theorem. However, this is admittedly far from being comprehensive, and we refer to the books Furstenberg (1981), McCutcheon (1999), and Einsiedler and Ward (2011) for more on this circle of ideas. For applications of ergodic theoretic techniques in *nonlinear dynamics* , see Lasota and Mackey (1994).

Apart from some notable exceptions, in this book we mainly treat the ergodic theory of a single measure-preserving transformation (i.e., $\mathbb{N}$ - or $\mathbb{Z}$ -actions). However, many of the notions and results carry over with no difficulty to measure-preserving actions of countable discrete (semi-)groups. For more about ergodic theory *beyond* $\mathbb{Z}$ *-actions* see Bergelson (1996), Lindenstrauss (2001), Tempelman (1992), Gorodnik and Nevo (2010) and the references given in Section 21.5.

Finally, the theory of *joinings* (see Thouvenot (1995) and Glasner (2003)) is not covered here. This theory has a strong functional analytic core and is connected to the theory of *dilations* and *disintegrations* of operators. Our original plan to include these topics in the present book had to be altered due to size constraints, and a detailed treatment is deferred to a future publication.

Notation, Conventions, and Peculiarities

For convenience, a list of symbols is included at the end. There, to each symbol we give a short explanation and indicate the place of its first occurrence in the text. At this point, we only want to stress that for us the set of *natural numbers* is

$$\mathbb{N} := \{1, 2, 3, \ldots\},$$

i.e., it does not contain 0, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is the set of nonnegative integers. (The meanings of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are as usual.)

For us, the word *positive* (number, measure, function, functional, etc.) means that the object under consideration is "≥ 0". That is, we call "positive" what in other texts might be called *nonnegative* .

In our definition of a *compact* topological space, the Hausdorff property is included, see Appendix A. Similarly, our notion of a *commutative algebra* includes the existence of a unit element, and algebra homomorphisms are to be understood as unital, see Appendix C.2.

History

The text of this book has a long and complicated history. In the early 1960s, Helmut H. Schaefer founded his Tübingen school systematically investigating Banach lattices and positive operators (Schaefer 1974). Growing out of that school, Rainer Nagel in the early 1970s developed in a series of lectures an abstract “ergodic theory on Banach lattices” in order to unify topological dynamics with the ergodic theory of measure-preserving systems. This approach was pursued subsequently together with several doctoral students, among whom Günther Palm (1976b), (1976a), (1978) succeeded in unifying the topological and measure theoretic entropy theories. This and other results, e.g., on discrete spectrum (Nagel and Wolff 1972), mean ergodic semigroups (Nagel 1973), and dilations of positive operators (Kern et al. 1977) led eventually to a manuscript by Roland Derndinger, Rainer Nagel, and Günther Palm entitled “Ergodic Theory in the Perspective of Functional Analysis,” ready for publication around 1980.

However, the “Zeitgeist” seemed to be against this approach. Ergodic theorists at the time were fascinated by other topics, like the isomorphism problem and the impact the concept of entropy had made on it. For this reason, the publication of the manuscript was delayed for several years. In 1987, when the manuscript was finally accepted by Springer’s Lecture Notes in Mathematics, time had passed over it, and none of the authors was willing or able to do the final editing. The book project was buried and the manuscript remained in an unpublished preprint form (Derndinger et al. 1987).

Then, some 20 years later and inspired by the survey articles by Bryna Kra (2006), (2007) and Terence Tao (2007) on the Green–Tao theorem, Rainer Nagel took up the topic again. He quickly motivated two of his former doctoral students (T.E., B.F.) and a former master student (M.H.) for the project. However, it was clear that the old manuscript could only serve as an important source, but a totally new text would have had to be written. During the academic year 2008/2009, the authors organized an international internet seminar under the title *“Ergodic Theory—An Operator Theoretic Approach”* (Figure 1) and wrote the corresponding lecture notes.

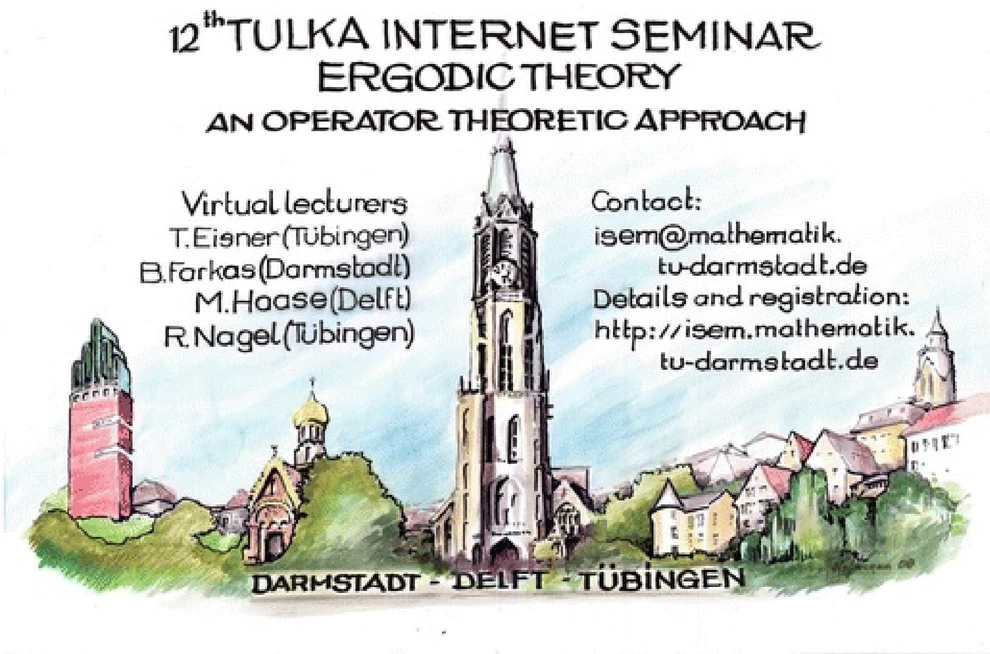

Fig. 1 The poster of the Internet Seminar by Karl Heinrich Hofmann

Over the last 6 years, these notes were expanded considerably, rearranged and rewritten several times (cf. the quote at the beginning). Until, finally, they became the book that we now present to the mathematical public.

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